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PROVIDING A FOUNDATION FOR DEDUCTIVE REASONING: STUDENTS' INTERPRETATIONS WHEN USING DYNAMIC GEOMETRY SOFTWARE AND THEIR EVOLVING MATHEMATICAL EXPLANATIONS

ABSTRACT. A key issue for mathematics education is how children can be supported in shifting from 'because it looks right' or 'because it works in these cases' to convincing arguments which work in general. In geometry, forms of software usually known as dynamic geometry environments may be useful as they can enable students to interact with geometrical theory. Yet the meanings that students gain of deductive reasoning through experience with such software is likely to be shaped, not only by the tasks they tackle and their interactions with their teacher and with other students, but also by features of the software environment. In order to try to illuminate this latter phenomenon, and to determine the longer-term influence of using such software, this paper reports on data from a longitudinal study of 12-year-old students' interpretations of geometrical objects and relationships when using dynamic geometry software. The focus of the paper is the progressive mathematisation of the student's sense of the software, examining their interpretations and using the explanations that students give of the geometrical properties of various quadrilaterals that they construct as one indicator of this. The research suggests that the students' explanations can evolve from imprecise, 'everyday' expressions, through reasoning that is overtly mediated by the software environment, to mathematical explanations of the geometric situation that transcend the particular tool being used. This latter stage, it is suggested, should help to provide a foundation on which to build further notions of deductive reasoning in mathematics.

KEY WORDS: appropriation of learning, computer environments, deductive reasoning, dynamic geometry software, geometry, mathematical explanation, mathematisation, mediation of learning, quadrilaterals, secondary school, sociocultural theory

INTRODUCTION

Providing a meaningful experience of deductive reasoning for students at the school level appears to be difficult. A range of research has documented that even after considerable teaching input, many students fail to see a need for deductive proving and/or are unable to distinguish between different forms of mathematical reasoning such as explanation, argument, verification and proof (for recent reviews see Hanna and Jahnke, 1996; Dreyfus, 1999). Amongst the reasons put forward for these student difficulties are

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that learning to prove requires the co-ordination of a range of competencies each of which is, individually, far from trivial (Hoyles, 1997), that teaching approaches often tend to concentrate on verification and devalue or omit exploration and explanation (de Villiers, 1998), and that learning to prove involves students making the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures (Dreyfus, 1999).

Despite the sheer complexity of learning to reason deductively in mathematics, and the wealth of evidence suggesting how difficult it can be for students, there are studies that show that students can learn to argue mathematically. Research by, for instance, Maher and Martino (1996) and Zack (1997) at the elementary school level, and by Dreyfus and Hadas (1996) and by Boero and colleagues (for example, Boero et al., 1996) at the high school level, illustrate how students can develop elements of deductive argument and how these notions of proving can depend on the classroom ethos, the tasks the students tackle, and the form of interactions that take place between the students and between the teacher and the students, as well as the tools available to the students. It is because of the complex nature of the interactions between these elements that, as Dreyfus (1999) concludes, much remains unknown about how students' mathematical deductive reasoning evolves and changes.

In geometry, an area of the curriculum intimately connected with the development of the deductive method, computer software packages generally known as dynamic geometry environments (DGEs) appear to have the potential to provide students with direct experience of geometrical theory and thereby break down what can be an unfortunate separation between geometrical construction and deduction (for a review see Chazan and Yerushalmy, 1998, p.72–77). As such, student use of a DGE could have an important role to play in enabling students to formulate deductive explanations and provide a foundation for developing ideas of proof and proving. Concerns remain, however, that the opportunity afforded by the software of testing a myriad of diagrams through use of the 'drag' function provided by the DGE, or of confirming conjectures through measurements (that also adjust as the figure is dragged), may *reduce* the perceived need for deductive proof (Chazan, 1993; Laborde, 1993; Hanna, 1998; Hoyles and Jones, 1998). If students are to gain facility with the deductive method through experience with a DGE then a particularly important issue is the impact that using such software has on the interpretation that students give to the geometrical objects they encounter in this way and how they learn to express explanations and verifications of geometrical theorems, properties and classifications (see Laborde and Laborde, 1995; Hoyles 1995).

Much previous research with dynamic geometry environments has quite properly focused on students in upper secondary (senior high) school where the students have received considerable teaching input in plane geometry, including the proving of elementary theorems, but may be new to the particular software tool. What is less clear at the moment is what impact the use of dynamic geometry software has on students in lower secondary (junior high) school where students have limited experience of the formal aspects of geometry but where contact with geometrical theory *through* the software may be especially valuable in providing a foundation for further work on developing deductive reasoning. At the moment too few students successfully make the transition to more formal mathematical study in which proof and proving are central. The aim of the study reported in this paper is to contribute to what is known about enabling more students to successfully make this transition.

In order to try to illuminate the impact of using such software, and to determine the longer-term influence, this paper reports an analysis of data taken from a longitudinal study (see Jones, 1996, 1997, 1998) of lower secondary (junior high) school students (aged 12 years old) learning aspects of geometry in a particular DGE, in this case *Cabri-géomètre* version 1.7 (Baulac et al., 1988). The focus of the paper is the progressive mathematisation (see below) of the students' sense of the software, reporting their interpretations and using explanations that the students give of the geometrical properties of the figures they construct as one indicator of this.

In this paper, an explanation is taken to be "that which explains, makes clear or accounts for" (see the Oxford English Dictionary, 1989) and verification is taken as "demonstration of the truth or correctness by fact or circumstances" (*op.cit.*). In terms of mathematics, and following Balacheff (1988a: 2), a mathematical explanation is taken as "the discourse of an individual intending to establish for somebody else the validity of a [mathematical] statement". In addition, and where these terms are used, a proof is "an explanation which is accepted by a community at a given time", and a mathematical proof is "a proof accepted by mathematicians" (again following Balacheff).

The analysis of the data from the longitudinal study that is reported in this paper focuses on the students' interpretations, especially the appropriation of mathematical terminology for explaining within geometric contexts as this is mediated through the dynamic geometry environment. As Edwards (1997, p. 188) explains, "in order to effectively support the teaching of proof with meaning, we must understand *how* students learn to reason, how they come to perceive and describe mathematical patterns, how generalizations and mathematical arguments are constructed, and how these processes can be supported in the learner" (emphasis in original). This paper concentrates on how the students reason about geometrical objects and relations as they experience them through the dynamic geometry environment and how the mathematical explanations they offer evolve as they become more experienced both with geometry and with the software.

STUDENT INTERPRETATIONS OF DYNAMIC GEOMETRY ENVIRONMENTS

In this section a brief outline is given of some of the research findings about how students interpret geometrical objects and relations when using dynamic geometry software. In general, as Goldenberg and Cuoco (1998) observe, as yet much remains unknown about how students glean geometric ideas from the complexly moving figures that they can encounter with a DGE. What is known is that the computer environment affects the actions that are possible when solving problems (a task solved using dynamic geometry software may require different strategies to the same task solved with paper and pencil) and it affects the feedback that is provided to the user (see Laborde, 1992, 1993a, 1993b). The DGE also introduces a specific criterion of validation for the solution of a construction problem: a solution is *valid* if and only if it is *not* possible to "*mess it up"* by dragging (to use the expression adopted by Healy et al., 1994, see also Noss et al., 1994), or, in other words, that there is "robustness of a figure under drag" (as used by Balacheff and Sutherland, 1994, p. 147). This criterion of validation does not depend on the perceptive appearance of the product of the construction as this appearance can be modified using the drag facility. To pass this 'drag test' the figure has to be constructed in such a way that it is consistent with geometrical theory.

Laborde (1993a, p.49) highlights an important distinction between *drawing* and *figure*: "*drawing* refers to the material entity while *figure* refers to a theoretical object". In terms of a dynamic geometry package, a *drawing* can be a juxtaposition of geometrical objects resembling closely the intended construction (something that can be made to 'look right'). In contrast, a *figure* additionally captures the relationships between the objects in such a way that the figure is invariant when any basic object used in the construction is dragged (in other words, that it passes the drag test). Hölzl (1995, 1996) found that learners can get 'stuck' somewhere between a drawing and a figure. He suggests that this relates to the fact that in a DGE such as *Cabri-géomètre* the verification process is controlled by the drag mode. He suggests that the more powerful the computer tool, the more didactic efforts are needed to provoke pupils to focus on the relevant mathematical relationships.

A further open question at the moment is how students can distinguish fundamental characteristics of geometry from features that are the result of the particular design of the DGE. As Balacheff (1996, p. 7) demonstrates with respect to *Cabri-géomètre*, the sequential organisation of actions necessary to produce a figure in *Cabri* introduces an explicit order of construction where, for most users, order is not normally expected or does not even matter. For example, *Cabri-géomètre* induces an orientation on objects: a segment AB is orientated because A is created before B. This probably makes sense in a direct manipulation environment, but it is contradictory to the fact that in paper-and-pencil these objects have no orientation – unless it is explicitly stated. In a complex figure this sequential organisation produces what is, in effect, a hierarchy of dependencies as each part of the construction depends on something created earlier. Hoyles (1995, p. 208) identifies this as a potential source of confusion in *Cabri-géomètre* as any hierarchy of relationships which has been established cannot then be modified (without undoing much that has been done, or even starting the whole construction again).

When observing young students attempting to construct a rectangle using the dynamic geometry software package *Cabri*, Hölzl et al. (1994, p. 11) found that, in order to make any progress with such a task, the students had to come to terms with "the very essence of Cabri; that *a figure consists of relationships* and that there is *a hierarchy of dependencies*" (emphasis in original). An example of this hierarchy of dependencies is the difference (in *Cabri*1 for the PC) between basic point, point on object and point of intersection. While all three types of point look identical on the screen, basic points and points on objects are moveable (with obvious restrictions on the latter). Yet a point of intersection cannot be dragged. This is because a point of intersection depends on the position of the basic objects which intersect. Only the basic objects (such as basic points, lines, etc.) used in a construction can be dragged. Dependent objects, such as points of intersection, only move *as a consequence of their dependency on these basic objects*. From their study, Hölzl et al. conclude that students need to develop an awareness of such functional dependency if they are to be successful with non-trivial geometrical construction tasks when using dynamic geometry software. Such an idea as functional dependency is, within the dynamic geometry environment, intimately connected with the notion of the robustness of a figure under drag mentioned above. Given the complexities involved, Hölzl et al. report that "not surprisingly, the idea of functional dependency has proved difficult [for students] to grasp" (*op.cit.*).

The above suggests that if DGEs are to be useful resources for helping to build a foundation for deductive reasoning for younger students then it is important to know what interpretations these students make of geometrical objects and relations experienced through a DGE. Of particular importance is their sense of 'drawing' and 'figure' (following the work of Laborde and of Hölzl) and of the nature of geometrical objects and relations, especially the notion of dependency. The next section gives a concise outline of the theoretical underpinning of the study reported in this paper.

THEORETICAL UNDERPINNING

In addition to a consideration of the previous research carried out with dynamic geometry software, some of which is referred to above, the theoretical framework used to locate and inform the design, implementation and analysis of the empirical component of the overall longitudinal study, from which the data reported in this paper is taken, is derived from research in the following areas:

a) Theoretical models of the teaching and learning of geometrical concepts, especially the van Hiele model (see, for example, van Hiele, 1986; Fuys et al., 1988),
b) Theoretical perspectives on the teaching and learning of deductive reasoning, especially the work of Balacheff (1988a and b), de Villiers (1990, 1998), and Hanna (1990, 1998),
c) Sociocultural perspectives on learning, especially the work of Wertsch (1991, 1998),
d) Theoretical perspectives on the role of technological tools in the learning process, especially the work of Pea (1987, 1993),
e) Theoretical perspectives on mathematisation, including the work of Gattegno (1988) and Wheeler (1982) together with, in a fairly restricted sense, aspects of mathematisation in work in the realistic mathematics education (RME) paradigm, for example, Treffers (1987).

This paper is mostly concerned with the latter theoretical component, the notion of mathematisation, which is dealt with in more detail below. The main features of the other theoretical positions, as they relate to this particular study, are as follows. The van Hiele model provides a background framing for the study, especially as transitions between the van Hiele levels are thought to be critical psychological shifts with important implications for further learning. For example, in the van Hiele model the shift from level 2 (identifying geometrical figures by their properties, which are seen as independent) to level 3 (recognising that a geometrical property of a figure precedes or follows from other properties), and subsequent progress within level 3, is especially important as these lay the foundations for full deductive proving in van Hiele level 4. The available evidence is that such progress made by school students is slow, with few students progressing beyond level 2, even by the end of secondary (high) school (Senk, 1989). In terms of the use of dynamic geometry software, an in-depth case-study of a single grade 6 student suggests that using such software may help students to progress to the higher van Hiele levels (Choi koh, 1999). When examining students' deductive reasoning and proving processes, Balacheff (1988b: 216–218) found it useful to distinguish between what he called "*pragmatic* proofs", which are "those having recourse to actual actions or showings", and "*conceptual* proofs", which "do not involve action and rest on formulations of the properties in question and relations between them". In terms of what is learnt and how it is learnt, from the perspective of sociocultural theory the assumption is that using a tool such as a dynamic geometry package does not serve simply to facilitate mental processes that would otherwise exist. Instead, use of the software is thought to fundamentally shape and transform the mental processes of the users (see Jones, 1998; Mariotti and Bartolini Bussi, 1998; and, for a more general discussion of semiotic mediation, Bartolini Bussi and Mariotti, 1999). Finally, following from this and using the terminology of Pea (1987, 1993), exploratory software environments in mathematics education, of which dynamic geometry is an example, can be said to act as cognitive *reorganisers* rather than merely amplifiers of existing human capabilities.

As noted above, this paper mostly uses (a somewhat specialised version of) the notion of mathematisation, and, indeed, *progressive* mathematisation (an idea taken from Treffers, 1987, see below). Mathematisation, following Gattegno (*op.cit.*) and Wheeler (*op.cit.*) involves a range of processes and facilities such as the ability to perceive relationships and to idealise them into purely mental material, the capacity to internalise actions and such like (so as to ask "What would happen if?"), and the ability to transform along a number of dimensions, such as from actions to perceptions and from images to concepts. Most often it is used when looking at a 'real' situation (as in the RME tradition), abstracting from it those elements that are wanted for closer study (or that appear to be tractable), setting a mathematical model, making inferences within the model, checking to see whether the results in the model are bourne out by observation and experiments, tinkering with the model to make it approximate reality better, and so on. Progressive mathematisation, as defined by Treffers (*op.cit.*), is the process whereby mathematical models are developed through the successive positioning of contexts that embody the underlying structure of the concepts.

These ideas of mathematisation, and progressive mathematisation, are used in this paper in the following sense. In tackling tasks involving elementary school geometry using dynamic geometry software, students can be said to be involved in modelling the geometrical situation using the tools available in the software. This involves setting up a construction and seeing if it is appropriate, and quite probably having to adjust the construction to fit the specification of the problem (producing a 'figure', rather then a 'drawing', in the sense of Laborde, above). For the purposes of this paper, this is taken as a form of mathematisation. When the tasks the students are tackling are linked in some way, such as involving the properties of quadrilaterals (with a focus on the relationship between such properties), this mathematisation process becomes more like a process of progressive mathematisation as the students develop a sense of the underlying relationships between the geometric properties.

In the next section the precise methodology is described. Overall, the research design follows the example of Meira in focusing on how "instructional artifacts and representational systems are actually used and transformed by students *in activity*" (Meira, 1995, p. 103, emphasis in original) rather than solely asking whether the students learn particular aspects of geometry *better* by using a tool such as a DGE when compared to using other tools (such as ruler and compass). The reason for this is that the focus of interest is both *what the students learn* and *how they learn it*.

RESEARCH DESIGN

The empirical work for this study was designed to be carried out in the UK and, following Hoyles (1997), the design was informed by the structure of the mathematics curriculum experienced by students in the UK. As Hoyles describes, while formal proof is likely to be restricted in the UK to the most able students only (and probably only encountered by students in upper secondary school), the curriculum does provide for opportunities for conjecturing and presenting generalisations at all levels. In terms of geometry, the curriculum attempts to incorporate aspects of the following geometries: plane, analytic/co-ordinate, and transformation. Such curriculum considerations mean that students in lower secondary (junior high) school typically know, for example, some of the properties of certain geometrical figures, have some experience of conjecturing and describing observations in open-ended problem situations, but have not been introduced to the formal aspects of proof and proving. In terms of experience with computer environments, students typically have followed an information technology course, which means they have some facility with using the computer mouse, menu items, and computer files.

The geometrical topic chosen as most suitable for the empirical study was the classifying of quadrilaterals. There were two main reasons for this decision. First, in terms of the 'family of quadrilaterals', there has been both discussion about the classification of quadrilaterals (for example, de Villiers 1994), and research involving students' ability to classify them, including both research using the van Hiele model (for instance, Fuys et al., 1988), and research involving pupils using the mathematical programming environment Logo (see Hoyles and Noss, 1992, for a comprehensive review). As de Villiers (1994, p. 11–12) explains, classifying is closely related to defining (and *vice versa*) and classifications can be hierarchical (by using *inclusive* definitions, such as a trapezium or trapezoid is a quadrilateral with *at least* one pair of sides parallel – which means that a parallelogram is a special form of trapezium) or partitional (by using *exclusive* definitions, such as a trapezium is a quadrilateral with *only* one pair of sides parallel, which *excludes* parallelograms from being classified as a special form of trapezium). In general, in mathematics, inclusive definitions (and thus hierarchical classifications) are preferred (although it should be stressed that exclusive definitions and partitional classifications are certainly not incorrect mathematically, just less useful). De Villiers (1994, p. 17) quotes from a number of studies (including Fuys et al., 1988) that have shown very clearly that many students have problems with the hierarchical classification of quadrilaterals. He suggests that some of the difficulties "do not necessarily lie with the logic of inclusion as such, but often with the *meaning* of the activity, both linguistic and functional: linguistic in the sense of correctly interpreting the language used for class inclusions, and functional in the sense of understanding why it is more useful than a partition classification". He observes that dynamic geometry software "offers great potential for conceptually enabling many children to see and accept the possibility of hierarchical inclusions (for example, . . . letting them drag the vertices of a dynamic parallelogram . . . to transform it into a rectangle, rhombus or square)". These comments from de Villiers, together with a reading of the other work on the classification of quadrilaterals, informed the precise form of the empirical work described below.

The second reason for choosing the classifying of quadrilaterals was more practical than epistemological. It was that one major strand within the geometry component of the UK mathematics curriculum is understanding and using the properties of plane shapes. In the lower secondary years (ages 11–13), the emphasis on this component of the curriculum moves from (informally) recognising and sorting geometrical figures towards the formal definitions required to classify and deduce properties of, and relations between, such figures. Choosing this topic meant that it fitted with the school's programme of study and no special dispensation would be needed to incorporate the topic (in the UK the curriculum is statutory and special dispensation is required if a school wishes to vary from it). Choosing a commonly occurring topic also means that the research findings may be found useful by practising secondary mathematics teachers.

Wherever possible, design choices were made with a view to the typicality of the setting. The school selected for the empirical work was an urban comprehensive school whose results in mathematics at age 16 were at the national average (there is a national system of testing in the UK that allows such judgements to be made). The mathematics teachers in the school used a problem-based approach to teaching mathematics and the students usually worked in pairs or small groups on mathematical problems and occasionally used computers. Throughout their mathematics work the students were expected to be able to explain the mathematics they were doing, either orally or in writing. This meant that work on geometry using dynamic geometry software would fit with the usual experience of the students. The classes of 12 year-olds in the school had four 50-minute mathematics lessons per week.

The particular class of 12 year-olds selected for the research were judged typical of that suitable for studying the relationships between quadrilaterals in that they were above-average in mathematics for their age (the school allocated students to different mathematics classes according to attainment in mathematics tests). A teaching unit was developed in collaboration with the teacher of the class that would address the properties of quadrilaterals and could be accommodated in the regular routine of the class. For most of the 9 months of the study up to four computers were available in the classroom. This meant that, as pairs of students took turns in using the computers, there might be gaps of up to a week between sessions that any particular pair of students had using the software. During these 'gaps' the students undertook other mathematics work, including some geometry topics involving area and volume, but not directly about the geometric properties of quadrilaterals. The version of *Cabri-géomètre* in use was *Cabri I* for the PC.

All the students in the class were tested using a van Hiele test (Usiskin, 1982) at the start of the unit of work and on its completion. The teaching unit was prepared to form three phases, and designed to fit around other

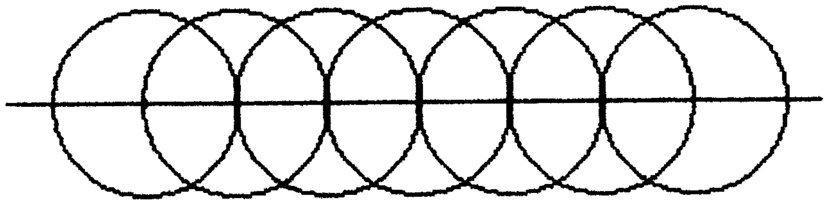

Figure 1. An example task from phase 1.

mathematics work for the class. During each of the phases, the students worked in pairs (usually the pairs they worked in for all their mathematics work).

During phase 1 the students gained preliminary experience with *Cabri-géomètre* while working through a short series of tasks involving lines and circles (see Figure 1 for an example). The aim of the phase was for students to acquire familiarity with the software interface and be introduced to the constraint of robustness of a figure under drag (see Balacheff and Sutherland, 1994, p. 147). For each task (in phase 1 and in the subsequent phases) the challenge for the students was to reproduce, using the software, a figure identical to one provided on paper but which could *not* be 'messed up' (this phrase suggested by Healy et al., 1994, was used consistently with the students). Figure 1 shows an example of one of the tasks from phase 1. The tasks used in each of the three phases of the study were so designed that to successfully meet the challenge of constructing the figures so that they are invariant under drag, the students have to analyse the spatial arrangements (taken as a form of mathematisation) *and*, as they are novices with the software, work out how to realise their constructions in the software environment in such a way that not only does it appear to be correct visually in a static form but that if any objects (such as points, lines or circles) used in the construction are dragged, the patterns remain consistent. Following Laborde (1993a, p. 49), the challenge for the students is to produce a 'figure' (which makes use of geometrical relationships) rather then a 'drawing' (which only looks like the required figure and which fails the dragging test of validity). For most students, phase 1 took up to three hours of using the software.

Phase 2 of the teaching unit involved the students working through a series of three tasks that required constructing the following quadrilaterals: a rhombus, a square, and a kite. Each task contained a visual prompt and the challenge to construct the figure so that it was invariant under drag and explain why the figure constructed is a particular quadrilateral. Figure 3

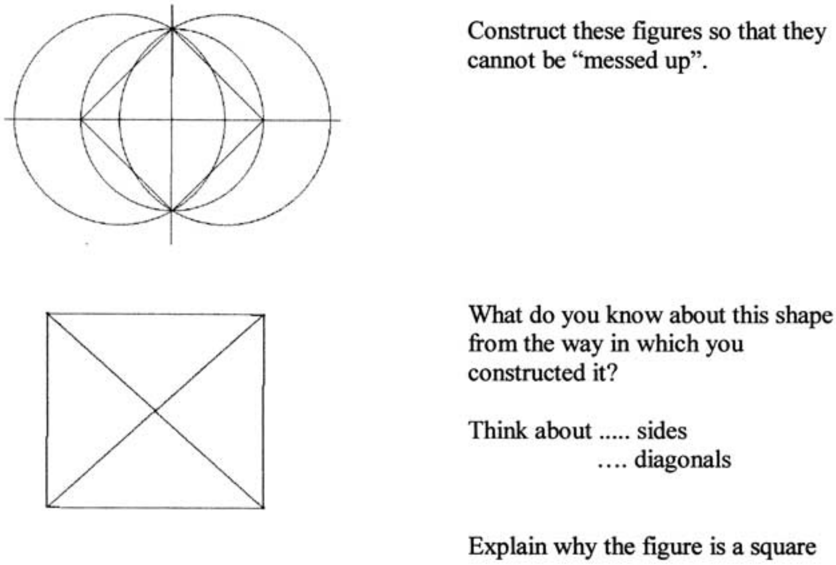

Figure 2. Visual prompt for constructing a square (task 2, phase 2).

shows the visual prompt the students received for constructing a square (task 2 in the phase 2 sequence of tasks).

In consonance with the idea of progressive mathematisation, the tasks in phase 2 were designed with the intention that the students would become more adept at analysing the geometrical structure provided in the visual prompt *and*, by having done this, be able to use this analysis to explain why the shape was the particular quadrilateral. The sequencing of the tasks (rhombus, square, kite) was designed with the intention that students might use approaches and geometrical properties gleaned from solving earlier tasks in approaching the later ones (including those in phase 3 below). While, initially, the notion of robustness under drag was introduced as a challenge to the students, both phase 2 of the teaching unit and phase 3 were designed so that this notion became intimately connected with the geometrical properties of the quadrilaterals being constructed. In this way it was intended that the students would come to appreciate, in terms of geometrical reasoning, the significance of robustness under drag and its usefulness as a test of the validity of constructions in terms of geometrical theory. The students took about two hours to complete phase 2 of the teaching unit.

Phase 3 of the teaching unit involved the students working through a series of six tasks that involved relationships between various quadrilaterals: the rhombus and the square, the rectangle and the square, the kite and the rhombus, the parallelogram and the trapezium, the rhombus, rectangle and the parallelogram. Most of the students took up to three hours on this phase of the teaching unit. For each task, the students were provided with a visual prompt similar to Figure 2 and the challenge to construct the figure so that it was invariant under drag and explain why all squares are rectangles, for example, or why all rectangles are parallelograms. The reason for this choice of phrase is to investigate the students' developing conception of this *inclusive* mathematical classification.

The sixth and final task of phase 3 asked the students to complete a hierarchical (inclusive) classification of the 'family' of quadrilaterals and explaining the relationships within this 'family' (see Figures 3a and 3b). In this final task the phrasing used was that a particular quadrilateral (say a square) was "a special case" of another quadrilateral (in this example, both of a rhombus and a rectangle). This phrasing was chosen as another way of expressing inclusive mathematical classification. Through the use of these different phrasings (that *all* squares are rectangles, and that a square is a *special case* of a rectangle) an attempt was made to take account of what Hershkowitz (1990, p. 81) calls "the opposing direction inclusion relationship" between sets and subsets of *examples* of, say, quadrilaterals, on the one hand, and the sets and subsets of their *attributes* on the other (for example, that the set of squares is included in the set of parallelograms, which, in turn, is included in the set of quadrilaterals, but the set of critical attributes of squares includes the set of critical attributes of parallelograms, which includes the set of critical attributes of quadrilaterals). The impact of this opposing direction inclusion relationship is that, for example, young children may not entertain a square as a quadrilateral because a square has four equal sides while other quadrilaterals do not.

During each phase, efforts were made to keep interventions by the teacher and the researcher to the following: responses (often in the form of questions) to student questions, and asking students for explanations. At times, suggestions were offered when students did not know how to proceed. The timing and nature of these occasions were noted. Particular attention was paid to instances when technical geometric terminology was introduced and used. The nature and impact of the interventions is an important aspect of the analysis of the data from this study and will be reported elsewhere. In planning the empirical study it was anticipated that the nature of the interventions would change during the various phases of the teaching unit, from ones concerned with aspects of the software in phase 1, to ones relating the properties of particular quadrilaterals in phase 2, and ones involving the relationships between quadrilaterals in phase 3. A conscious decision was made to focus the interventions on the geometrical properties of the quadrilaterals and not to encourage use of the measuring tools (such as length and angle) available in the software environment. Following the theoretical framework and the usual practice of the class teacher, the nature of the interventions was designed to be consonant with guided participation in sociocultural activity. As noted above, the norm of the classroom was for students to provide explanations and for these to be related to the structure of the problems being tackled.

There were 28 students in the experimental class. Seven pairs of students were studied during phase 1 in order to select four pairs for detailed study during phases 2 and 3. The four pairs of students studied during phases 2 and 3 were selected both to represent the range of attainment in the class in terms of the van Hiele levels and on the basis that each pair worked reasonably well together. The four pairs were as follows:

- Pair A: both students van Hiele level 1–2
- Pair B: one student van Hiele level 1–2, the other level 2
- Pair C: both students van Hiele level 2
- Pair D: one student van Hiele level 2, the other level 2–3

For all the pairs of students studied, the following data was collected: video and additional audio tape to capture the onscreen work and student-student and student-teacher interactions, student written work (unaided), student software files, the 'history' of the student constructions using the software (a feature available with the particular software), and researcher field-notes.

RESULTS AND ANALYSIS

The analysis presented below focuses on data from two pairs of students, pairs A and C, chosen because the individuals in each pair were assessed as being at similar van Hiele levels. According to the van Hiele model, as van Hiele explains, "each [van Hiele] level has its own linguistic symbols and its own relations connecting these symbols. A relation which is 'correct' at one level can reveal itself to be incorrect at another. Think for example, of a relation between a square and a rectangle. Two people who reason at different levels cannot understand each other. Neither can manage to follow the thought processes of the other" (van Hiele, 1959; as quoted in Fuys et al., 1988, p. 6). Thus choosing pairs A and C in this paper means that the analysis can focus more precisely on the interpretations and explanations of the students without being over-complicated by possible misunderstandings *between* the students. In the overall research design, pairs B and D were chosen in order to examine these inter-student aspects in more detail, particularly when one student is more capable (in terms of the van Hiele levels) than the other. These inter-student aspects are subject to separate analysis. Suffice to say for this paper that the overall pattern of development analysed below is typical of that found with all the pairs of students.

In the analysis the term 'mathematical reasoning' is used to denote logical inference and deduction of a form appropriate to lower secondary (junior high) school students (and should not be taken to mean the use of abstract symbolic notation, truth tables or formal axiomatic proofs, for example). In the context of this study mathematical reasoning can involve using the properties of shapes to judge the validity of results and justifying steps in giving explanations for statements. Precision in mathematical reasoning is taken to include the use of the mathematical terms that are appropriate for lower secondary school students, as opposed to 'everyday' terms for geometrical objects such as 'oblong', 'diamond' or 'oval'. Thus 'mathematical reasoning' is taken to mean making reasonably precise statements and deductions about properties and relationships. Such reasoning can be quite detailed without necessarily being thorough enough to be called a proof.

The first part of this section reports on phase 1 of the teaching unit, focusing in particular on how the students interpreted the notion of the constraint of robustness of a figure under drag. The second part of the section reports on the two pairs of students as they work through a series of tasks that involved constructing various quadrilaterals. The focus is their interpretations using their evolving mathematical explanations as an indication of this as they attempt to explain the properties of the quadrilaterals they construct.

Phase 1 of the teaching unit

Phase 1 of the teaching unit served to introduce the students to the software and to the constraint of robustness of a figure under drag. In this phase the students worked through a short series of tasks involving patterns of interlinking lines and circles (see Figure 1 for an example).

The data from this phase raises several issues about the interpretation the students made of the software environment. Some of these were fairly trivial. For example, the distinction between lines (that are infinite) and line segments – a distinction that these particular students had not met previously. Other aspects of the software environment took longer for the students to become accustomed to. The most important were:

1. The aspect of functional dependency as realised in the software whereby some objects can be dragged while others cannot.

2. The distinctive properties of points of intersection (which, in *Cabri 1*, had to be explicitly created).
3. The sequential organisation required in order to construct a figure that is robust under drag.

These three aspects relate to phenomena created by the software (and only relate to geometrical theory in terms of how this theory is realised in the software) and it is crucial that students recognise this if they are going to see *through* the software to the geometry.

For the students in the study reported in this paper, the notion of the constraint of robustness of a figure under drag became linked with using points of intersection to try to hold the figure together. To illustrate this, the extracts below are from the transcribed sessions and written work of pair C (pseudonyms Heather and Karol). In all the data extracts presented below, square brackets are used to insert short phrases in order to clarify the meaning of the extracts. Normal use is made of question marks and exclamation marks. Short pauses are shown by a series of dots.

On two occasions during session 1 (of phase 1), the students raised questions and received input from the teacher about dependency and about points of intersection. On the first occasion the students want to delete a point. When attempting to do so they get the following message from the software: 'Delete this object and its dependents?'. They ask the teacher what this means. The teacher suggests that they go ahead and delete the point and see what happens (after reassuring them that they can undo the delete). The students delete the point and two line segments disappear. This gives the teacher an opportunity to explicitly refer to dependency:

Teacher: that bit of line depended on that point, and that bit of line did, so they both went.

[Pair C, session 1 (phase 1)]

Later in the session the students discover that the points of intersection they have constructed cannot be dragged. Here the teacher again refers to dependency and explains how points of intersection depend on the other objects. After a little thought and dragging, one of the students says:

Karol: You can't drag that point [a point of intersection] because it is dependent on them [indicating the points used to create the shape].

[Pair C, session 1 (phase 1)]

The other student nods, which appears to indicate that, at this point, the students appreciate what is different about points of intersection and how this relates to dependency. During the next session, however, there is evidence that they have developed their own, somewhat different, interpretation.

In session 2, while tackling another task involving lines and circles, the following exchange takes place:

Karol: What's the [point of] intersection doing? Does it keep the dot [the point] there?
Teacher: What you are finding is the point *here*, where the circle crosses the line.
Karol: Right, so if it was like *that* [indicating a different arrangement of lines], then it [the point of intersection] would be *there*.
Teacher: It is always where the lines cross.

[Pair C, session 2 (phase 1)]

In raising the question about points of intersection, here is a first indication of the interpretation of the students have of points of intersection. It seems that Karol may think that such points have a role in 'holding' a figure together so that it is invariant under drag. Later in the session the other student gives another indication:

Heather: You have to make an intersection between those two lines so that they can't be moved.

[Pair C, session 2 (phase 1)]

In this case it is not altogether clear what the student means. However, the students do successfully complete the tasks for the session and, at the end of the session, are asked why their figures cannot be 'messed up'. Heather replies:

Heather: They stay together because of the intersections.

[Pair C, session 2 (phase 1)]

This statement is not incorrect as it relates to their perspective on robustness under drag, but it also masks another interpretation that is revealed in the next session. In their third session, the students are in the process of constructing a rhombus, which they need to ensure is invariant when any basic point used in its construction is dragged. As they go about constructing a number of points of intersection, one of the students comments spontaneously:

Heather: [referring to a point of intersection] a bit like glue really. It just glued them together.

[Pair C, session 3 (phase 2)]

This spontaneous use of the term ‘glue’ to refer to points of intersection has been observed by other researchers (see, for example, Pratt and Ainley, 1996) and is all the more striking given the fact that earlier on in the lesson the students had confidently referred to such points as points of intersection (implying that such points only exist if other geometrical objects intersect).

A little later in the same session, one student in the pair again asks the teacher why it is not possible to ‘drag’ points of intersection. The teacher replies:

Teacher: Because the intersection points just show you where two things cross.

Karol: So how come it keeps it together if it’s just a dot to show you where they cross?

Teacher: You can move that point because it’s the centre of the first circle that you drew. So if you move that [point], then because you are changing the size of the first circle, the point where it crosses the other circle changes so that changes the other circle.

Karol: So that changes everything.

Teacher: Because the other circle depends on that.

Heather: So because it depends on it, it moves.

[Pair C, session 3 (phase 2)]

Here one of the students, Karol, asks, “So how come it [a point of intersection] keeps it [the figure] together. . .?”. The teacher does not address this directly but returns to the idea of dependency.

At the end of each session the students are expected to write down something about the session. Below is an extract from what this pair writes at the end of the following session:

> Things are ‘dependent’ and when they are made dependent they can’t move. Dependence is created by an intersection between two things. For this exercise we made everything dependent on the perpendicular line so although things move in different ways, one thing holds everything in place.
> You have to make them dependent on each other, or another object, so it stays how we want it to.

[Pair C, session 4 (phase 2)]

In writing that things “can’t move” the student is employing their own description of invariance under drag. Clearly, under drag, things do move, including points of intersection. The ‘things’ that do not ‘move’ are the geometrical relationships that have been constructed (parallel lines remain parallel, for example). It is the hierarchies of dependencies, and the sequential organisation required by the software (see Balacheff, 1996, p. 7), that the students are beginning to appreciate are central to constructing figures in a DGE (such that any particular construction is robust under drag). These are features specific to the software and the way that geometrical theory is realised in that environment. As noted above, Hölzl et al. (1994) found that students in their study needed to develop an awareness of such functional dependency if they were to be successful with non-trivial geometrical construction tasks using a DGE. Their experience was that this idea was difficult for students to grasp. The students in this study similarly took time to grasp the idea of functional dependency. As also noted above, the spontaneous use of the conception of points of intersection as 'glue', see student comment from session 3 (phase 2), is a phenomenon observed by other studies of younger students' early experiences with *Cabri* (see, for example, Pratt and Ainley, 1996). It occurred spontaneously in this study reported, despite the planned interventions by the teacher that focused on the notion of dependency. Further evidence of the difficulty students have with interpreting points of intersection is provided by Hoyles (1995, pp. 210–211).

The above examples illustrate how the notion of the constraint of robustness of a figure under drag became linked with using points of intersection to try to hold the figure together. The focus for the students was on the mechanical aspects of the software environment, as illustrated by their attempts to use points of intersection to ensure that their construction could not be 'messed up', rather than on the geometry of the figure being constructed. These interpretations of the software environment made by the students in phase 1 (and into phase 2) of the teaching unit is an important aspect of the progressive mathematisation of the students' sense of the software environment. It provides part of the basis by which the students began tackling the tasks in phases 2 and 3, which involve geometrical theory much more explicitly.

In the analysis below, of phases 2 and 3, a major source of data is the unaided writing of the student pairs that they produced during and at the end of each session. This is augmented by extracts from the transcribed recordings of the student oral explanations. The reason for focusing on the student explanations is to reveal how they progressively mathematise the sense they made of the software environment and how this impacts on their developing mathematical reasoning. Such a focus means, however, that, in general, there is little space to show *how* the students came to their particular explanations. This is not the focus of this paper. Data from the overall corpus is selected for each of phases 2 and 3 to illustrate the main aspects. The selection has been made on the basis of the representativeness of the data in illustrating the overall trend of the students' progressive mathematisation.

Phase 2 of the teaching unit

Task 1 of phase 2 asked the students to construct a rhombus and explain why the shape is a rhombus (the task is similar in format to that for constructing a square as shown in Figure 2). The explanations of the students are as follows:

Pair A written explanation:

> The radius is the same for the circle and the diamond [the rhombus] and we made the diamond from the help of the first construction. The sides are all the same because if the centre is in the right place the sides are bound to be the same. The diagonals of the diamond cross in the middle though they are different size (length). They cross at the middle through the line. Their diagonals bisect each other. The angles [at the intersection of the diagonals] are all the same. They are 90°. The opposite angles [of the rhombus] are the same. Two are more than 90° but less than 180° and the others are less than 90° but more than 0°.
> This shape is a rhombus because the sides are the same, the diagonals bisect at right angles and the opposites have the same angles.

Pair C, in their written explanation, concentrate on recording their understanding of dependency (see above). Below is an extract from the session transcript that took place near the end of the session:

Teacher: What sort of shape is that?
Karol: It's a rhombus.
Teacher: How do you know it's a rhombus?
Karol: Our old maths teacher used to call a rhombus a drunken square, because it's like a square, only sick.
Teacher: What do you know about a rhombus, from what you have done?
Heather: It's got a centre.
Karol: It's like a diamond . . .. But it's not a square.
Teacher: What can you say about the sides or the angles . . . or the diagonals?
Karol: Those two angles [indicating the angles at one pair of opposite vertices] are the same, and those two are the same . . . [indicating the other pair of opposite angles]
But they are not all the same [indicating adjacent angles]
And . . . the sides are all the same length . . . I think.
Heather: It's the same distance across each side.
Teacher: What can you say about how the diagonals cross?

Karol: A right angle.
Teacher: How do you know?
Karol: It looks straight.

These outcomes are essentially descriptive rather than explanatory, and there is evidence of a lack of student capability with precise mathematical terminology (viz. the use of 'diamond' by both pairs) and some reliance on perception rather than mathematical reasoning (for example, "It looks straight"). As the students are familiar with the properties of a rhombus they are able to provide these. However, they are likely to be unfamiliar with the notion of *economic* mathematical definitions (de Villiers, 1994, p. 12), that is, definitions that contain only necessary and sufficient properties, and with how the properties are related (beyond writing, for example, that "if the centre is in the right place the sides are bound to be the same").

In task 2 of phase 2, the students are asked to construct a square and explain why the shape is a square (see Figure 2). The written explanations of the students are as follows:

Pair A written explanation:

> It is a square because the sides are equal and the diagonals intersect. The diagonals are [at] right angles (90°).

Pair C written explanation:

> It [the square] is made up of four equal sides. Its diagonals are equal. We know the diagonals are equal because they are the diameters of the circle. The diagonals cross at 90 degrees. The diameters have to be equal for it to be a circle, and [the] diagonals have to be equal at 90 degrees [for it] to be a square.
> It is a square because two equal diagonals cross each other at a 90 degree angle.

For this task, the written work of pair A remains essentially descriptive while pair C (the more able pair) include an explanation that the diagonals of the square are equal "because they are the diameters of the circle". For both pairs, their use of mathematical terminology is more precise than in task 1.

Phase 3 of the teaching unit

While the three tasks in phase 2 (on which data from two is provided above) were concerned with constructing individual quadrilaterals, the tasks in phase 3 involved constructing a specified quadrilateral (for example, a rectangle) in such a way that by dragging one of the vertices it could be modified (or transformed) into a special case (in the example of a rectangle, the special case would be a square).

Task 5, of the overall sequence of tasks on quadrilaterals (the first task in phase 3), required the construction of a rectangle that could be modified to a square. The students were then expected to explain why all squares are rectangles. These are their written explanations:

Pair A written explanation:

> A rectangle . . . becomes a square when the diagonals become right angles where they meet.

Pair C written explanation:

> You can make a rectangle into a square by dragging one side shorter and so the others become longer until the sides become equal.

These are very reasonable explanations of how a rectangle can "become a square". As can be seen from the students' explanations, these are couched in terms of the nature of the software environment, pair C more overtly (through use of the term "dragging"). Both pairs write about something *becoming* something. These 'pragmatic' explanations, it is proposed, are analogous to the notion of 'pragmatic' proofs, being ones that have "recourse to actual actions" (Balacheff, 1988b, pp. 216–218).

What is not evidenced here, at this point, is whether the students appreciate that, using inclusive definitions, a square is a special case of a rectangle (which they may feel is different from a rectangle being able to "become" a square) or, similarly, that all squares are rectangles (even though they were asked about that). Partly, of course, this is the result of the task, yet the task depends on the software. In the DGE, by definition, a quadrilateral constructed as a square cannot be modified to a non-square as the drag test of validity means that it must remain a square whatever basic objects used in its construction are dragged.

Task 6 required the students to construct a kite that could be modified to a rhombus and explain why all rhombi are kites. In task 7 the students are asked to construct a trapezium that can be modified to a parallelogram and thereby explain why all parallelograms are trapeziums. These are their written explanations:

Pair A written explanation:

> It is a trapezium because it has one pair of parallel lines. A parallelogram is parallel both ways.

Pair C written explanation:

> Trapeziums have one set of parallel lines and parallelograms have two sets of parallel lines.

Neither of the students' written explanations is couched in terms of the nature of the software environment. There appears to have been a shift from 'pragmatic' explanations that reflect the nature of the software environment to mathematical explanation. The cause of this shift is mainly due to the role of the teacher in consistently referring to the geometrical properties of the shapes whenever there was an interaction with the students. The role played by the teacher and the impact of the teacher-student interactions is the subject of separate analysis.

Task 9 asked the students to complete the worksheet shown in Figure 3a as a way of showing the relationships between the 'family' of quadrilaterals

All the pairs of students completed this task satisfactorily, in each case with some interaction with the teacher. This interaction consisted of the teacher referring the students back to earlier tasks they had completed in the teaching unit, and the teacher asking the students to explain any relationship between the various quadrilaterals that the students could identify. A completed worksheet (from pair A) is given in Figure 3b. Below are extracts from the transcripts of the lessons relating to this task involving pairs A and C.

Extracts from Pair A session transcript (pseudonyms Harri and Russell):

Teacher: Why is a square a special sort of rectangle?

Russell: Because they've both got right angles [at the vertices] but with a rectangle [indicating one that is not a square] one of the sides is bigger than the other.

Teacher: Why is a rectangle a special case of a parallelogram?

Harri: The two opposite [sides] are the same length but [indicating a parallelogram that is not a rectangle] they [the angles at the vertices] are not right angles.

Extracts from Pair C session transcript (pseudonyms Heather and Karol):

Teacher: Why is a square a special sort of rhombus?

Heather: Because in a square all the . . . all the corners are 90 degrees and all the sides are equal, but in a rhombus [indicating one that is not a square] all the sides are equal but they [the angles at the vertices] are not 90-degree angles.

Teacher: Why that arrow? [indicating that a rhombus is a special form of parallelogram].

Karol: It's just like the rhombus and the square because . . . because all the sides . . . the sides are . . . the opposite sides are of equal length, but there [in the

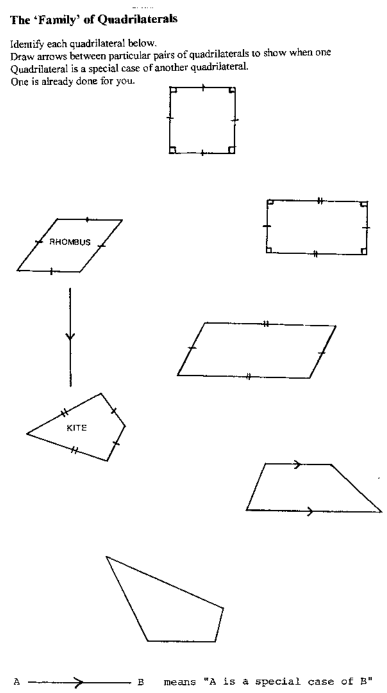

Figure 3a. Worksheet on the 'family' of quadrilaterals.

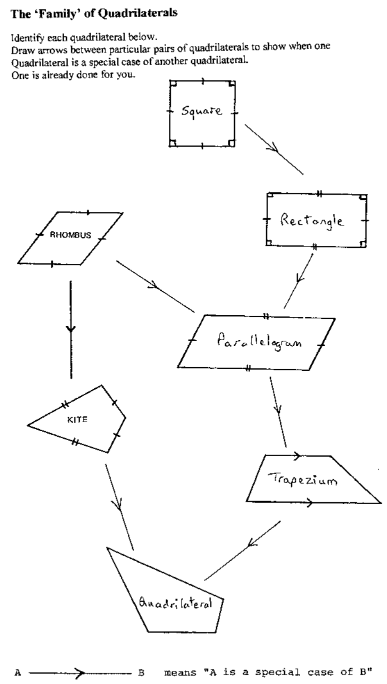

Figure 3b. Worksheet on the 'family' of quadrilaterals.

> rhombus] they [the diagonals] cross at 90 degrees and there [indicating a parallelogram that is not a rhombus] they don't.

Thus, by the end of the teaching unit, the students were able to accept questions of the form 'why is one quadrilateral a special case of another' and could offer explanations as to why this is the case. These explanations are interesting in terms of how difficult it was for the students to articulate their explanations without recourse to concrete illustrations. Consider the explanation given by Russell in response to the question, why is a square a special sort of rectangle. Saying that "they've both got right angles [at the vertices] but with a rectangle [indicating one that is not a square] one of the sides is bigger than the other" could be taken as incorrect, mathematically, because, as the set of rectangles contains the set of squares (using an inclusive classification), it is incorrect to say that a rectangle "has one of the sides bigger than the other" (since the statement has to refer to squares too, for which it is patently untrue). The same argument can be made about the explanations offered by the other students. Each of them had to find a way of indicating that the set of more general quadrilaterals to which they were referring excluded the special case. This is because, as de Villiers (1994: 13) explains, partitioning is a "spontaneous and natural strategy" and that "we would normally call a square a 'square' and reserve the term 'rectangle' only for a non-square (or general) rectangle".

In explaining why a square is a special sort of rectangle, Russell could have made use of a term such as 'oblong' for those rectangles that are not squares. Yet this term is not only superfluous in an inclusive classification but it depends itself on an *exclusive* definition, something that it not always helpful from a mathematical perspective and is often discouraged in mathematics curricula and textbooks. The alternative would have been for the student to have used a much more complicated sentence structure, something like, "both squares and rectangles have right angles at their vertices but rectangles that are not squares have one of the sides bigger than the other". This is where issues to do with the logic of inclusive definitions and classification, the language used for class inclusions, and the functional aspects of hierarchical classification all have a bearing.

Discussion

The classroom used for this study was selected because the students were accustomed to a pedagogical approach involving mathematisation. The usual approach of the teacher was to have the students working in pairs or small groups (of 3 to 6 students) on a range of problem-based tasks, some 'real' (and similar to those in the RME tradition) and some of a more 'pure' mathematical nature. The tasks the students tackled in the teaching unit developed for this study was of the latter form. Their progressive mathematisation over the period of the study can be summarised as follows:

- Initially, an emphasis on description rather than explanation. Some reliance on perception rather than mathematical reasoning. Lack of capability with precise mathematical language (similar to that found in other studies, for example Fuys et al., 1988, pp. 135–136).
- At an interim stage, explanations become more mathematically precise but are influenced (mediated) by the nature of the dynamic geometry software (for example by the use of the term 'dragging' or by other phrases linked to the dynamic nature of the software).
- At the end of the teaching unit, explanations related entirely to the mathematical context.

Overall, as the students worked through the teaching unit, there was a shift in their thinking from imprecise, 'everyday' expressions, through reasoning mediated by the software environment to mathematical explanations of the geometric situation.

Given the significant problems that many students have with the hierarchical classification of quadrilaterals (see, for example, Fuys et al., 1988 and de Villiers, 1994) the evidence reported in this paper supports the suggestion by de Villiers (*op.cit.*: p. 17) that dynamic geometry software offers "great potential for conceptually enabling many children to see and accept the possibility of hierarchical inclusions". Such an outcome should help to lay a solid foundation on which to develop further notions of deductive thinking.

The research study reported in this paper also reveals the mediational impact of using dynamic geometry software. As documented by this study (and by other research referred to in this paper), this mediational impact was in terms of the following:

- The students' understanding that the order in which objects were created leads to a hierarchy of functional dependency within a figure.
- The constraint of robustness of a figure under drag becoming linked with using points of intersection to try to hold the figure together.
- The 'dynamic' nature of the software influencing the form of explanation given by the students.

Thus, when using dynamic geometry software, students need to come to terms with the notion of a hierarchy of functional dependency within a figure (see, for further examples, Hölzl et al., 1994; Jones, 1996). Secondly, the students need to gain an appreciation of the notion of the constraint of robustness of a figure under drag as a *mathematical* feature, rather than, say, as 'mechanical glue' (Pratt and Ainley, 1996; Jones, 1998). Thirdly, the 'dynamic' nature of the software influences the form of explanation given by the students (what Hölzl, 1996, p. 184, refers to as reasoning "in a *Cabr*i-specific style").

As Hölzl (1996) points out, this latter example of tool mediation is akin to the notion of 'situated abstraction' proposed by Noss and Hoyles (1996, pp. 122–125) as a step in constructing a mathematical generalisation. In this conceptualisation, the abstraction is 'situated' in that the knowledge is defined by the actions within a context. Yet it is an abstraction in that the description is not a routinised report of action but exemplifies the students' reflections on their actions as they strive to communicate a mathematical explanation. Recall that Pair C wrote: "you can make a rectangle into a square by dragging one side shorter and so the others become longer until the sides become equal". Such an explanation, using Balacheff's (1988) distinction, could be called a *pragmatic* explanation in that it refers to actual actions. The form of explanations given by the students later in the teaching unit, in resting on the mathematical properties in question, would, to continue the analogy, constitute *conceptual* explanations.

The evidence from this study indicates that using dynamic geometry software does provide students with access to the world of geometrical theorems but it is access that is mediated by features of the software environment, certainly in the vital early and intermediate stages of using the software. The research described in this paper illustrates that with carefully designed tasks, sensitive teacher input, and a classroom environment that encourages conjecturing and a focus on mathematical explanation, students can make progress with formulating mathematical explanations and coming to terms with inclusive definitions, both important aspects of developing a facility with deductive reasoning. Without such factors, the mediational impact of the software could be such that it may distract students from the geometry of the problem situation or possibly *reduce* the perceived need for deductive proof.

De Villiers (1998) suggests that a focus on mathematical explanation is part of the route to a greater appreciation of the role and function of mathematical proof. This study was designed to illuminate the impact of dynamic geometry software on one small component of this, the relationships between quadrilaterals to form a hierarchical classification. While the study was informed by research on the van Hiele model of thinking in geometry, the results should not be taken as evidence, necessarily, of the validity of the van Hiele model. According to Duval (1998), for instance, a model of mathematics learning in which different ways of mathematical reasoning are organised according to a strict hierarchy (as in the van Hiele model) is inappropriate. Rather than being representative of higher (or lower) levels of thinking, Duval argues that different kinds of cognitive activity have their own specific and independent development (this may account for evidence that students can appear to be operating at more than one van Hiele level simultaneously, a finding reported by Gutiérrez, Jaime and Fortuny, 1991).

In the mathematics classroom, the practical issues of when and how to use dynamic geometry software are very important. Much previous research with dynamic geometry software has focused on students in upper secondary school where the students have received considerable teaching input in plane geometry, including the proving of elementary theorems, but are new to the particular software tool. The study reported in this paper focuses on students in lower secondary school where students have quite limited experience of the formal aspects of geometry (and have certainly never seen a proof or been asked to prove a theorem). The evidence presented in this paper confirms that students can make progress towards *mathematical* explanations, which, it is suggested, should provide a foundation on which to build further notions of deductive reasoning in mathematics.

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