General Frameworks for Mathematization and van Hiele Levels

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elated Articles

Key Questions and Contents

1. How does mathematical abstraction differ from other kinds of abstraction in its nature, in the way it develop?

   How does mathematician develop mathematics?

   How can we teach mathematics based on the processes?

2. How can we model the developmental processes for students?

For modeling the processes of abstraction;

Freudenhal's meanings of mathematization

General Framework of van Hiele Levels

Applying the model;

Levels of Functional thinking up to the Calculus

For Describing the Processes without Levels;

The processes of Mathematization from the view point of Mathematical Representation.

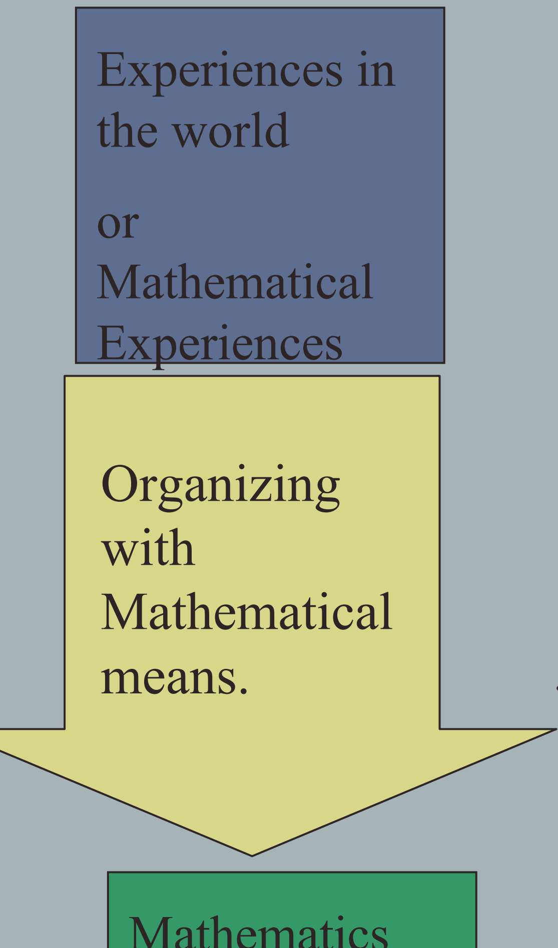
Dialectic Respecting the Nature of Mathematics

How does mathematical abstraction differ from other kinds of abstraction in its nature, in the way it develop?

Anna Sfard(1991)

- *From the view point of progressive mathemaization, Gravemeijer & Doorman (1999) descried;*
  - Freudenthal (1971) express the process of mathematization as 'the operational matter on one level becomes a subject matter on next level'. Although Freudenthal has *micro* levels in minds, a connection can be made with Sfard's(1991) more *macroscopic* account of mathematical development based on historical analyses.
- *Freudenthal (1994) himself did not accepted the idea of progressive mathematization;*
  - For a long time I have hesitated to accept the distinction of horizontal and vertical mathematization.
- *And re-defined the progressive mathematization as his mathematizaion;*
  - *Horizontal* mathematiz*(s)action* leads from the world of life to the world of symbols. In the world of life one lives, acts; in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflecting; this is *vertical mathematization*. The word of life is what is experienced as reality, as is symbol world with regard to its abstraction.
  - The distinction between horizontal and vertical mathematizing depends on the specific situation, the person involved and his environment. Apart from these generalities, examples on various levels are the best way to explain the difference between horizontal and vertical mathematizing.

Re-visiting Freudenthal's Mathematization (1973)



- *As soon as science outgrows mere collecting, it becomes involved in the organization of experiences. It is not difficult to indicate the experiences that should be organized in arithmetic and geometry. Organizing the reality with mathematical means is today called mathemtizing. The mathematician, however, is inclined to disregard reality as soon as the logical connection promises faster progress. A stock of mathematical experience is formed; it asks for its part to be organized. What kind of means will serve this purpose? Of curse, mathematical means again. This starts the mathematizing of mathematics*

Freudenthal proposed Van Hiele Levels as general framework for mathematization in school mathematics;

The learning process is structured by levels. The activity of the lower level, that is the organizing activity by the means of this level, become an object of analysis on the higher level; The operational matter of the lower level becomes a subject matter on the next level.

1. Students explore matter (object) using *figures (method)*.
2. Students explore the figures using the *properties*
3. Students explore the properties of figures using *implication*.
4. Students explore the proposition, which is formed by implication, using *proof*.
5. Students explore the proof, which formed by intuitive logic, using formal *logic*.

Generalization of van Hiele Levels

Stoliar (1969), Hoffer (1983), etc.

Hoffer generalized the levels with the idea of Categories

1. *Objects are the base elements of the study.*
2. *Object are properties that analyze the base elements.*
3. *Object are statements that relate the properties.*
4. *Object are partial orderings (sequences) of the statements.*
5. *Object are properties that analyze the partial orderings.*

On the other hands, many general frameworks only focused on that the method of activity is the object of next activity and lost the idea of levels.

What kind of ideas were lost only focusing on it?

The ways to apply the generalized levels to other area in mathematics

1. We expect some areas as the conceptual domain for levels.
2. Tentative description of levels based on the analogy of the idea of levels is constructed; the method of activity is the object of next level.
3. Illustrating the levels with phylogenetic and ontogenesic evidences; comparing with historical development, analyzing the curriculum and students development based on the curriculum.
4. Confirming the features of van Hiele Levels to recognize the levels as van Hiele Levels.
   - A) Language Hierarchy. (van Hiele, 1959).
   - B) Existence of Un-translatable Conceptions. (van Hiele, 1986).
   - C) Duality of Object and Method. (van Hiele 1958; H. Freudenthal 1973; I. Hirabayashi 1978).
   - D) Mathematical Language and Student Thinking in Context. (van Hiele, 1958; M. Isoda, 1988; D.Clements, 1992; cf. M. Battista, 1994).

Generalization of van Hiele levels from Geometry to Calculus *Isoda 1985*

with the analogy of the levels of geometry.

	The Levels of Geometry	The Levels of Function
Level 1	Students explore <u>matter</u> (<u>object</u>) using *figures* (*method*).	Students explore <u>phenomena</u> (<u>object</u>) using obscure *relations* or *variation* (*method*).
Example of conflicts between levels	Because it has rounded corners, the road sign 'YIELD' is not a triangle according to the meanings of Level 2, but we call it a triangle in daily language.	In Japanese, we use "2 BAI, 3 BAI" to mean "two times, three times" on level 2. But in everyday Japanese (Level 1), we can use "BAI" to mean either "double" or "plus". A child on level 1 says "BAI,BAI" ("plus plus") to mean three times the original amount. But "BAI,BAI" ("double double") usually means four times. On level 2, students use "2 BAI, 3 BAI" to explain proportion as a covariance and they say three times as "3 BAI" and do not say it "BAI,BAI".
Level 2	Students explore the <u>figures</u> using the *property*; The <u>object</u> on level 2 was the method on level 1.	Students explore the <u>relations</u> using *rules*; The <u>object</u> on level 2 was the *method* on level 1.
Example of conflicts	A square is rectangular on Level 3, but not on Level 2.	The constant function is a function on Level 3 but 'constant' is not the relation which was discussed as covariation on level 2.
Level 3	Students explore the <u>properties</u> of figures using *implication*.	Students explore the <u>rules</u> using *notations of functions*.
Example of conflicts	The isosceles triangle has congruent angles. On Level 3, it is induced already and we do not have to explain more. On Level 4, we prove it.	On Level 3, a tangent line of quadrilateral function deduced using the property of only one common point/ a multiple root. On the Level 4, the tangent line does not always have this property.
Level 4	Students explore the proposition, which is formed by implication using proof	Students explore <u>functions</u> using *derived* or primitive function

Levels of Function up to Calculus; Level 1 and Level 2

Language Hierarchy.

Existence of Un-translatable Concepts.

Duality of Object and Method.

Level 1 Everyday Language	Students explore **phenomena** (objects) using immature *relations* or *variation* (method).	Students describe relations in real world phenomena using everyday language immaturely. They can discuss changes in numbers using calculations, but usually their descriptions are done with or focused on one physically evident variable, the dependent variable. Even if they are aware of covariation, it is difficult for them to explain it appropriately using two variables because their descriptions of relations are done immaturely using everyday language. So it is difficult for them to compare different phenomena.
Example of conflicts between Level 1 and Level 2	In Japanese, we use "2 BAI, 3 BAI" to mean "two times, three times" on level 2. But in everyday Japanese (level 1), Japanese use "BAI" to mean any of "double", "plus" or Òmore Ó. Eg. ÒHito (person) Ich Bai (one time)Ó means Òmore than other personÓ or Òtwo times the other personÓ in Japanese usage. A child on level 1 says "BAI, BAI" ("plus, plus") to mean three times the original amount. But some students think "BAI, BAI" ("double, double") means four times. On the other hand, at level 2, students have to use "2 BAI, 3 BAI" to explain proportion as a covariance and they say three times as "3 BAI", not as "BAI, BAI".	
Level 2. Arithmetic Language	Students explore **relations** using *rules*. The **object** on Level 2 was the *method* on Level 1.	Students describe the rules for relations using tables. They make and explore tables with arithmetic. Their descriptions of relations in phenomena are more precise with tables than just with the everyday language of level 1. Students have general concepts about some relations, for instance, proportion. *Students can compare different phenomena using such rules.* They describe rules for relations as covariation and when reading tables, their interpretation of the covariation of variables is at least as strong as their interpretation of correspondence. Students begin to use algebraic formulas and graphs to represent rules and relations in phenomena but it is difficult for them to translate between notations without any phenomena.

Levels of Function up to Calculus; Level 2 and Level 3

Language Hierarchy.

Existence of Un-translatable Concepts.

Duality of Object and Method.

Level 2. Arithmetic Language	Students explore **relations** using *rules*. The **object** on Level 2 was the *method* on Level 1.	Students describe the rules for relations using tables. They make and explore tables with arithmetic. Their descriptions of relations in phenomena are more precise with tables than just with the everyday language of level 1. Students have general concepts about some relations, for instance, proportion. Students can compare different phenomena using such rules. They describe rules for relations as covariation and when reading tables, their interpretation of the covariation of variables is at least as strong as their interpretation of correspondence. Students begin to use algebraic formulas and graphs to represent rules and relations in phenomena but it is difficult for them to translate between notations without any phenomena.
Example of conflicts	The constant function, y = constant for any x, is a function on level 3 but 'constant' means no relation in the phenomena on level 2 because students on level 2 try to find a relation of covariational change in the phenomena.	
Level 3 Algebraic or Geometric Language	Students explore **rules** using *function notation*.	Students describe functions using equations and graphs. To explore functions, even where there is no reference to real world phenomena, they translate among *the notations of tables, equations and graphs and use algebra and geometry.* At this level, their notion of function, which they already understand well, involves the representation of different notations already integrated as a mental image. For example, they can easily find the equation corresponding to a graph, and a graph from an equation.

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Levels of Function up to Calculus; Level 3 and Level 4

Language Hierarchy.

Existence of Un-translatable Concepts.

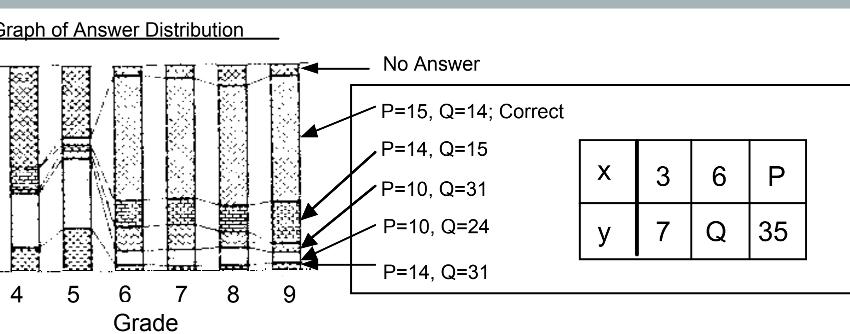
Duality of Object and Method.

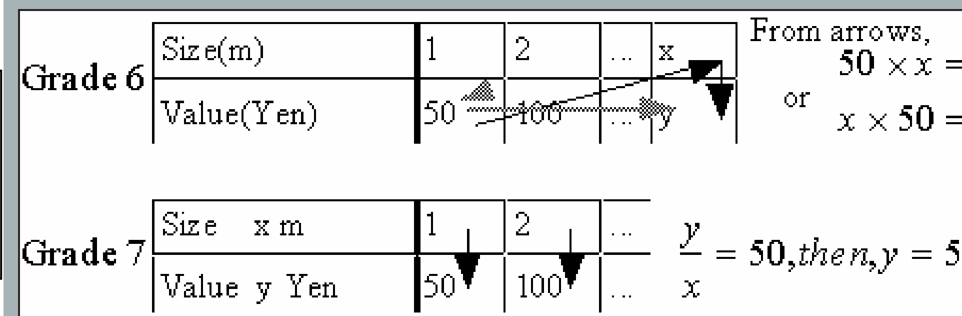
Level 3 Algebraic or Geometric Language	Students explore **rules** using *function notation*.	Students describe functions using equations and graphs. To explore functions, even where there is no reference to real world phenomena, they translate among *the notations of tables, equations and graphs and use algebra and geometry*. At this level, their notion of function, which they already understand well, involves the representation of different notations already integrated as a mental image. For example, they can easily find the equation corresponding to a graph, and a graph from an equation.
Exa mp l e o f con flic ts	On level 3, algebraically, a tangent line of a quadratic function can be deduced using the property that there is only one common point (ie. a multiple root). On level 4, using calculus, the tangent line does not always have this property.	
Level 4 Calculus with algebraic or geometric notation	Students explore **functions** using the *derived* or *primitive function*.	Students describe functions using calculus. In calculus, functions are described in terms of derived or primitive functions. For example, to describe the features of a function we use its derived function (which has already been learned). The theory of calculus is a generalized theory of such descriptions.

Different ways of thinking between Level 1 and Level 2

Isoda, 1989

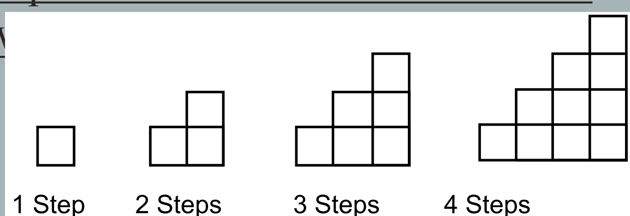
Problem 1 In the right table, if y is in proportion to x, then select the pair which is appropriate for P and Q in the table.

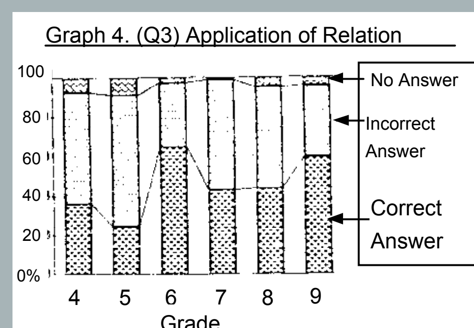




Problem 2. Let's make stairs using squares with sides 1 cm as follows

Q3. V ?





Different ways of thinking between Level 2 and Level 3

Isoda, 1989

Problem 3. Write what you can find from the following tables.

(1)

x	1	2	3	4
y	4	8	12	16

(2)

x	1	2	3	4
y	2	8	18	32

Result of (1)

Grade	6	7	8	9
Covariation	42%	48%	49%	35%
Correspondence	24%	14%	16%	35%
Both	8%	10%	11%	11%

Result of (2)

Grade	6	7	8	9
Covariation	27%	15%	11%	10%
Correspondence	37%	21%	20%	50%
Both	0	0	2%	3%

Implications

Illustrating the differences based on the difference of language;

Level.4

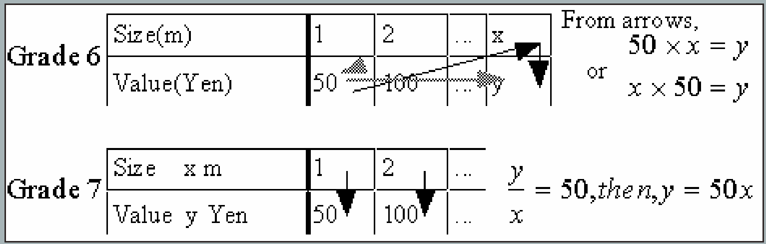
Problem 1. If we define the growth of a microorganism as proportional to the amount X of a fungus at time t, find the differential equation.

dx/dt=kx

Problem 2. If we define the growth of a microorganism as proportional to the amount X of a fungus at time t and *at the same time as proportionally decreasing depending on how 'close' to a maximum quantity Xmax it reaches, find the differential equation.*

dx/dt=kx(1-x/xmax)

Level.2



Level.3

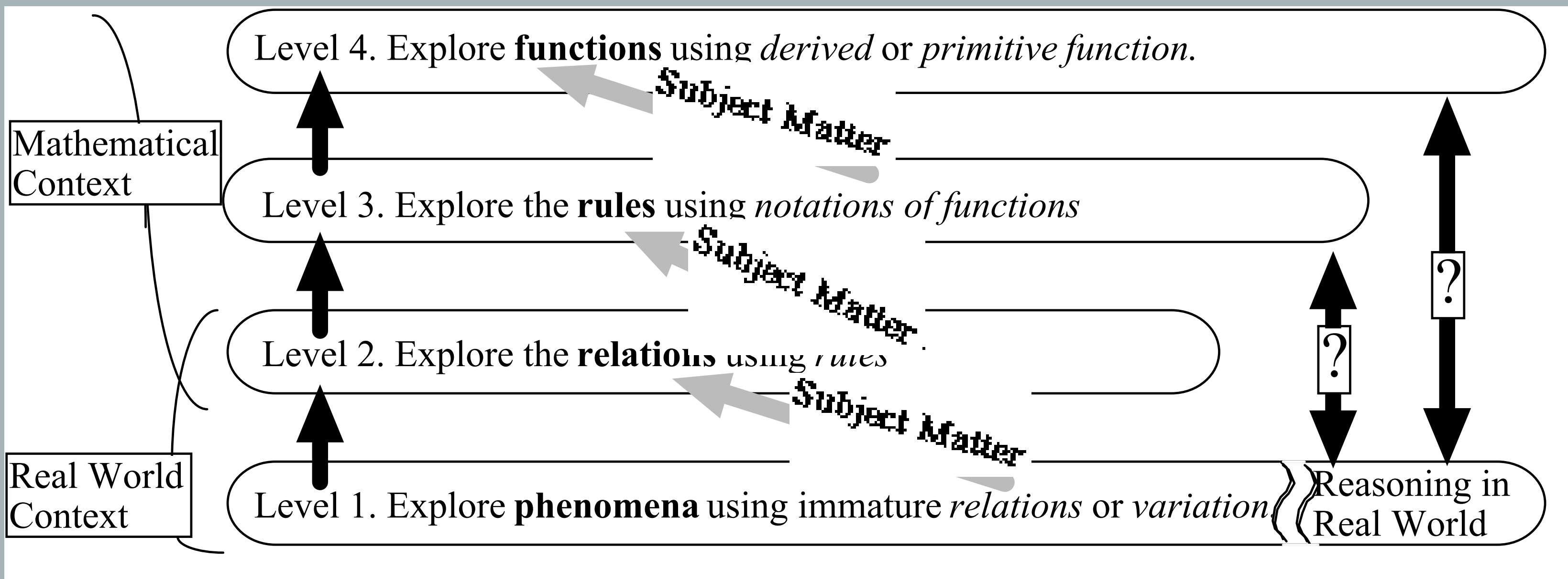
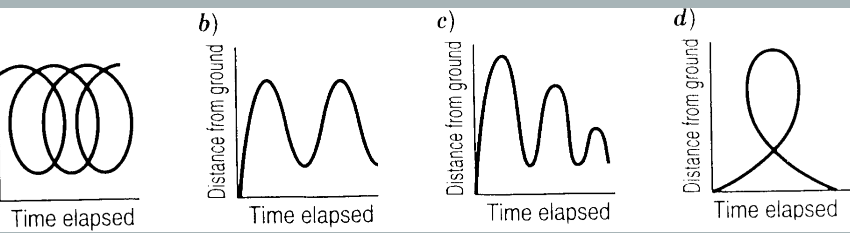


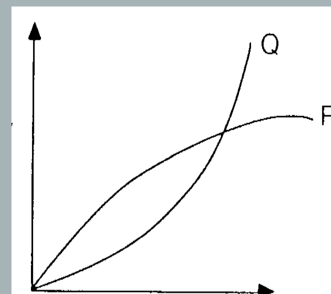
Figure 2. The development of levels does not guarantee applicability

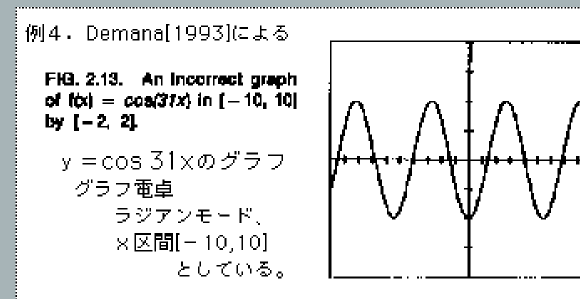
Implications

Illustrating the difficulties based on the levels;

Why the Achievement of both problems are very low?
What are differences?







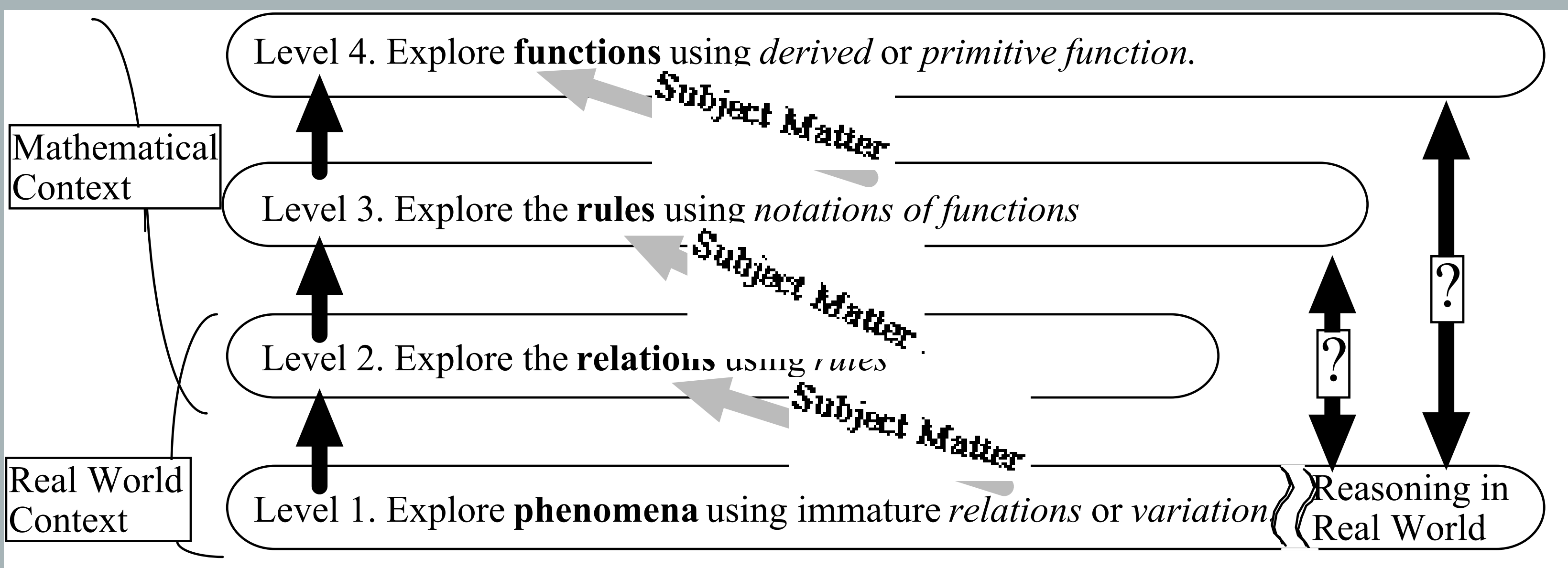


Figure 2. The development of levels does not guarantee applicability

Mathematical Representation

(ISODA 1991)

Elements of mathematical representation in thinking processes;

Symbol, Operation and Aim(or Context)

Mathematical Representation as notation system;

$3x=6$	$2x+3=5x-3$
$2x+3=5x-3$	$3x=6$
$x=2$	$x=2$
Symbol, Operation	and Aim(or Context)

Representation is a element of Representation System; R(Symbol,Operation),
Representation World is an Integrated Representation Systems depending on the situations or problems; W={Ri(Si, Oi) ⟺ Rk(Sk, Ok); …..}

<u>Problem</u>. There is a rectangler that the width is 3cm longer than the lengthwidth. We make another rectangler whose width is three times as long as the width of the based rectagler and whose lengthwidth is two times as long as the lengthwidth of the based rectagler. Then, the perimeter of the made rectagler is 10cm longer than two times as large as the perimeter of the based rectangler. How long is the perimeter of the based rectangler?

Problem. There is a rectangler that the width is 3cm longer than the lengthwidth. We make another rectangler whose width is three times as long as the width of the based rectagler and whose lengthwidth is two times as long as the lengthwidth of the based rectagler. Then, the perimeter of the made rectagler is 10cm longer than two times as large as the perimeter of the based rectangler. How long is the perimeter of the based rectangler?

width x, lengthwidth x-3

3x 2(x-3)

2{3x+2(x-3)}=2(x-3)+10

Then, x=5/3 ?

Width x, lengthwidth x-3

3x 2(x-3)

2(5x-3)=2{2(2x-3)}+10

Then, x=2?

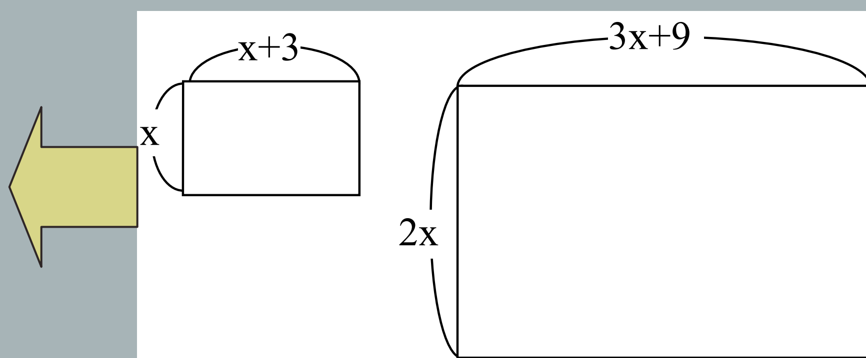
Width x+3, lengthwidth x

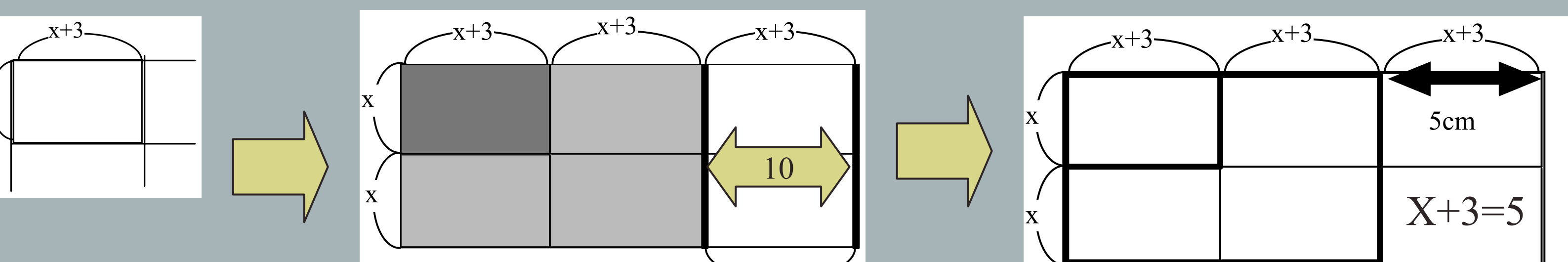
3x+9 2x

2(5x+9)=2{2(2x+3)}+10

Then, x=2

2(2x+3)□14, Ans. 14cm





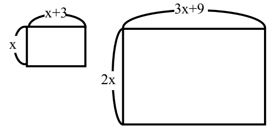
Nature of Mathematization from the viewpoint of Representation; From Non-Operational to Operational Representation (ISODA 1991)

Reasoning with image based on oneÕs experience
(at Real Word or Existent Level)

Problem and experience

width x, lengthwidth x-3
□@□@x 2(x-3)
2{3x+2(x-3)}=2(x-3)+10
Then, x=5/3 ?

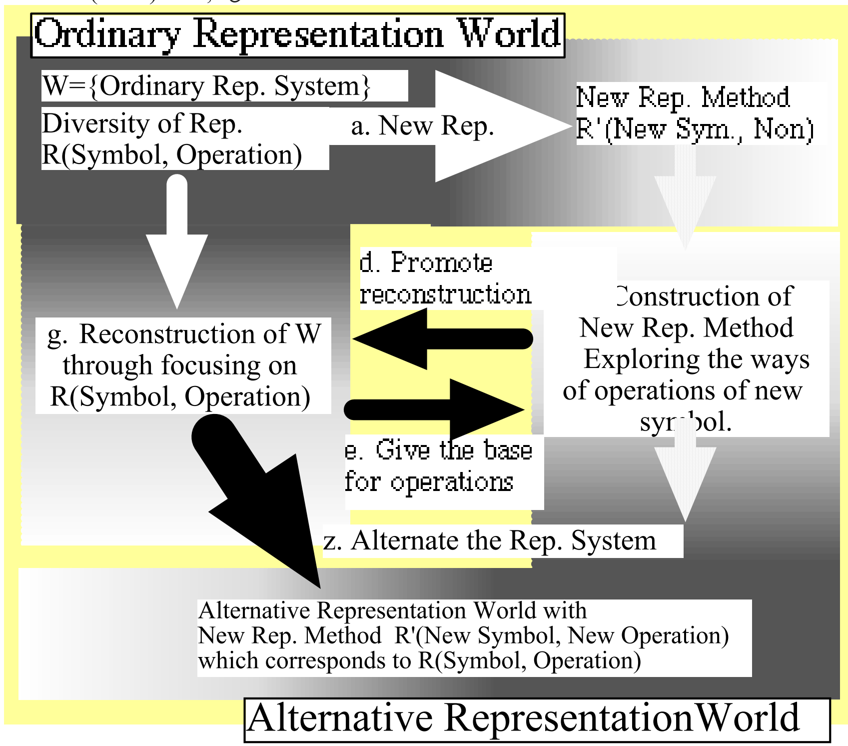
Width x+3, lengthwidth x
3+9 2x
2(5x+9)=2{2(2x+3)}
Then, x=2
2(2x+3)□□14,□@Ans. 14cm



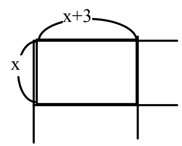
Width x, lengthwidth x-3
□@□@x□@□@□@□2(x-3)
2(5x-3)=2{2(2x-3)}+10
Then, x=2?

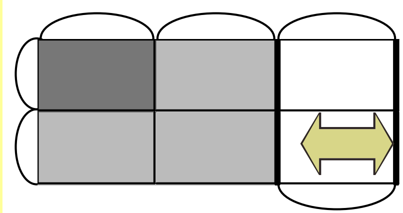
Reasoning but focused on the special structure at Real Word or Existent Level

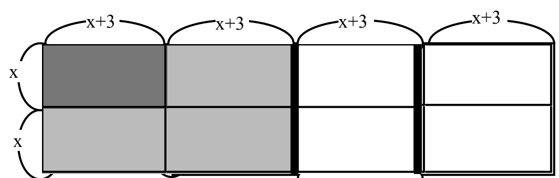
X+3=5



Developing the Reasoning with the mathematical representation without the special structure at Real Word or Existent Level







2(X+3)=10

Solvable Problem Structure

Reasoning with the mathematical structure and the mechanical structure (Synchronization in Alternative Level)

Modeling the processes of Mathematization from the view point of Mathematical Representation
ISODA, 1991

Ordinary Representation World; "In the world of life one lives, acts"

Reasoning with the image based on one's experience at Real Word or Existent Level

Reasoning with ordinal representation. New representation is introduced with the translation of ordinal representation. It could be operated with the translation of ordinal representation and it does not have autonomy as representation.

In the process of mathematization in the context of developing operation for new representation, autonomy as representation, following mutual interactive activity is ongoing;

Reasoning but focused on the special structure at Real Word or Existent Level

The representation which could be translated to the new representation is focused because of necessary to develop the operation of new representation.

Developing the Reasoning with the mathematical representation without the special structure at Real Word or Existent Level

The operation of new representation is developed with the translation of focused representation.

Alternative Representation World; "In the world of life one lives, acts"

Reasoning with the mathematical structure with Synchronization in Alternative Mathematical World or Level

After the development of the operation of new representation, the representation is used autonomicaly and the alternative representation world is integrated with it.

A Case Study; Applying the model of Representation Explore the Motion of Crank Mechanism

- 4 hours in high school, 9 female students who know trigonometric functions, group activity.
- The example illustrate the difficulty to develop synchronization.

In the process of mathematization in the context of developing operation for new representation, autonomy as representation, following mutual interactive activity is ongoing;

Reasoning but focused on the special structure at Real Word or Existent Level

The representation which could be translated to the new representation is focused because of necessary to develop the operation of new representation.

Developing the Reasoning with the mathematical representation without the special structure at Real Word or Existent Level

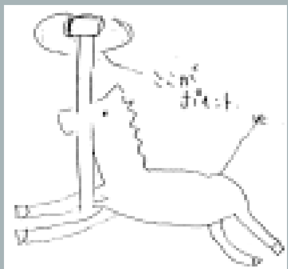
The operation of new representation is developed with the translation of focused representation.

Alternative Representation World; *"*In the world of life one lives, acts"

Reasoning with the mathematical structure with Synchronization in Alternative Mathematical World or Level

After the development of the operation of new representation, the representation is used autonomicaly and the alternative representation world is specified on the situations and integrated with it.

In a Daily Context; Reasoning with Visual Image Based on One's Experience



- *How does the wooden-horse of merry-go-round move?*
  - *Students do not understand how the circle motion produces an up-and-down motion.*

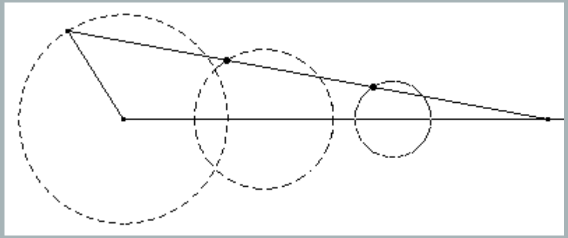
- *Make the mechanics by LEGO which could represent the motion of wooden-horse.*
  - *Students could construct only the separate parts.*

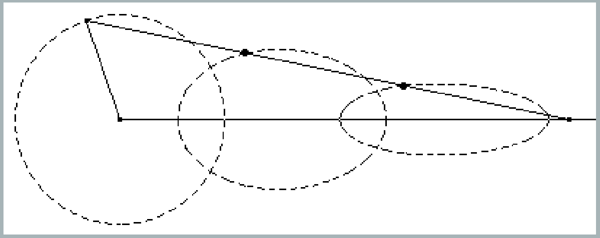
- Their images are too far from the cognitive structure to formulate the mathematical model

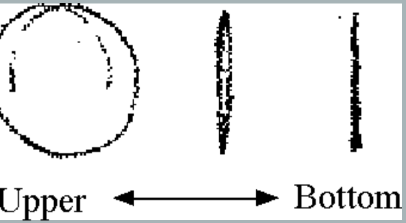
Conflict between Visual Image and the Locus by the Mechanics; Getting Structure of Mechanics Beyond Visualized Materials

- *Students made their crank using a sample by Teacher.*

  - *Students imaged that the locus of wooden-horse must be circle. They were still reasoning with their visual images and could not reason with mechanical structure even if they made the mechanics.*







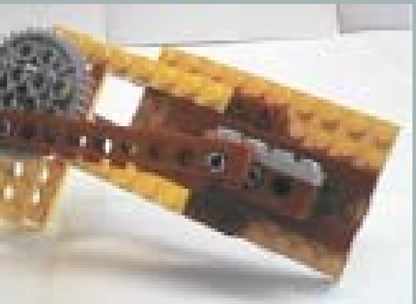
♦Students drew loci using the crank.

- 'I wonder that upper side is circle, but bottom side is a pressed oval and the height is same.'

♦*Students can over come misunderstanding by reasoning with the structure*

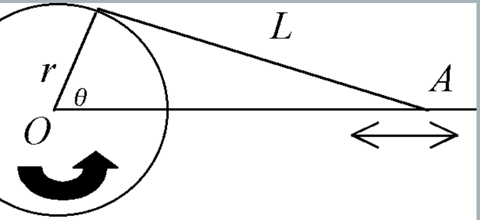
Formation of the Mathematical Model and Interpretations

Reasoning via Mathematical Model with Weak Mechanical Structure



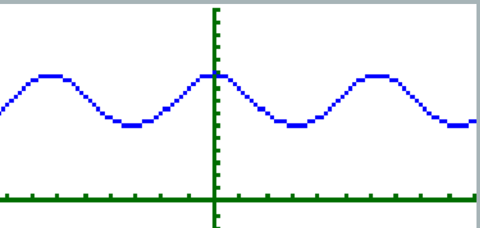
- *Students were asked to represent mathematically the up-and-down motion of the crank's piston, endpoint A, as an extreme case of the locus as the pressed ovals*

  □Students couldn't solve and teacher helped them.



$$f(\theta) = \mathrm{OA} = r\cos\theta + \sqrt{L^2 - 4\sin^2\theta}$$

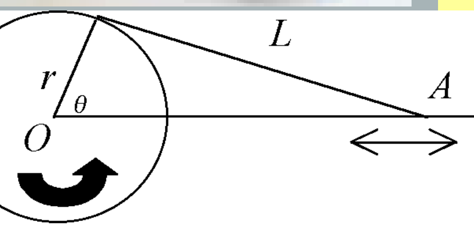
Teacher asked students to explore the meaning of the function with graphic calculator.

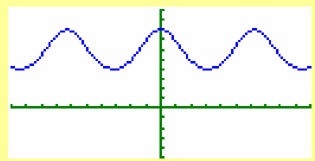
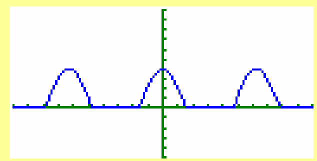
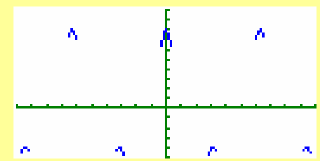


- Students compared the up-and-down motion of the piston via the LEGO crank with the graph of the function. "If the piston moves up, the cogwheel rotate right. And if the piston moves down, bar moves also down."
- It looks like they could success mathematical modeling

Changing the parameters of model did not mean changing the parts of mechanics□*Knowing the correspondence between the parameters of mathematical model and the parts of mechanical structure.*

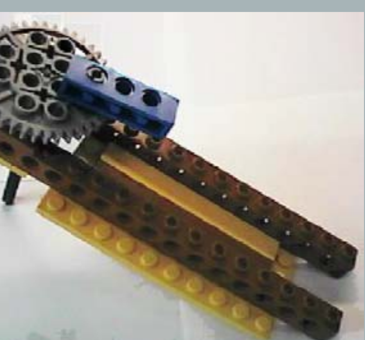
- *Students explore the mathematical model with graphic calculator through making a lot of problems via changing the parameters of function.*



Ratio	1) $r : L = 1 : 3$	2) $r : L = 1 : 1$	3) $r : L = 3 : 1$
Graphs			

□*From the case 2&3, students thought that the equation is wrong.*

- *Students tried to reproduce the case 2&3 by LEGO.*
  - □ *' If we apply the conditions of 2) or 3) to the wooden-horse will hit the cogwheel. These conditions are not appropriate for the crank mechanism. The length of L should be longer than of r for the crank (?)'*



Illustrating the model with the example from LEGO project

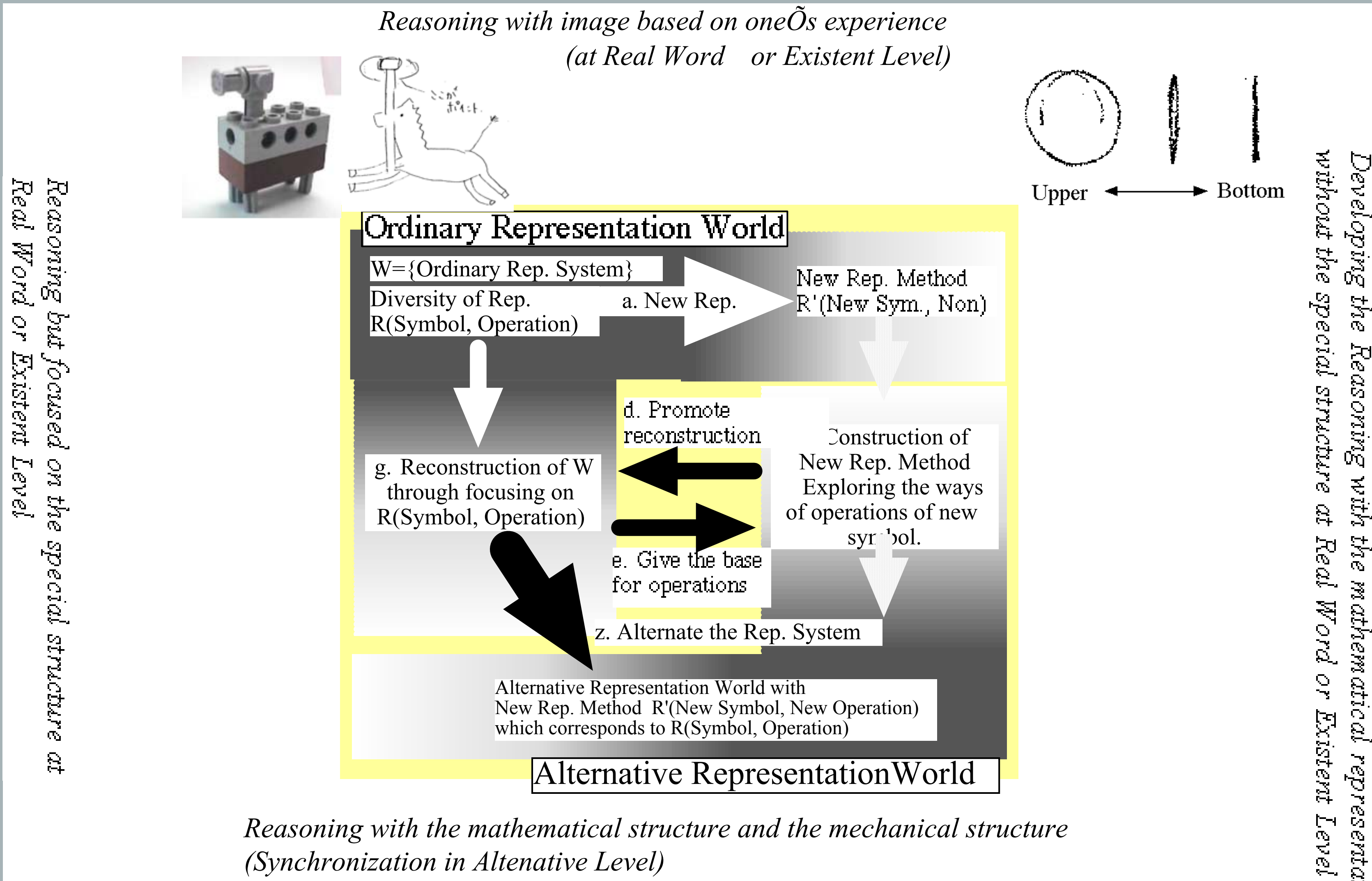


Figure.4 The Process of Mathematization from the view point of a representational

The Model of The processes of Mathematization from the view point of Mathematical Representation Significances and Restrictions

The Model illustrates the process of abstraction.

Reflective abstraction consists in deriving from a system of actions or operations at a lower level, certain characteristics whose reflection (in the quasi-physical sense of the term) upon actions or operations of a higher level it guarantees.

Reflective abstraction proceeds by reconstructions which transcend, while integrating, previous construction (Piaget, 1966)

The Model illustrates the abstraction is not normative.

Alternative representation world is restricted on the special situations and not always abstract in normative meanings.

The Model restricted the meaning of levels.

A) *From Language Hierarchy to The different worlds of Mathematical Representation.*

B) Existence of Un-translatable Concepts.

C) Duality of Object and Method.

D) Mathematical Language and Student Thinking in Context.

Related Articles written in English

- Isoda, M.(1996), The Development of Language about Function: An Application of van Hiele's Levels, Edited by Luis Puig and Angel Gutierrez, Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, Volume 3,105-112
- Isoda, M., Matsuzaki, A., (1999), Mathematical Modeling in the Inquiry of Linkages Using LEGO and Graphic Calculator: Does New Technology Alternate Old Technology?, Edited by W. Yang, D. Wang, S. Chu & G. Fitz-Gerald, Proceedings of the Forth Asian Technology Conference in Mathematics, ATCM Inc. USA, 113-122
- Isoda, M., (2001), Synchronization of Algebraic Notations and Real World Situations fromthe Viewpoint of Levels of Language for Functional Representation, edited by Chick, H., Vincent, J., Stacy, K.,The Future of the Teaching and Learning of Algebra: Preproceedings of the 12th ICMI conference, vol.1,