Symbolic Reasoning and Transformational Reasoning and Their Effect on Algebraic Reasoning

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Symbolic Reasoning

A Historical Note

During the nineteenth century an enormous amount of work was done in England in differential and difference calculus using a technique called “operational method”. In this method, results are obtained by symbol manipulations without understanding their mathematical justification, and in many cases they even violated well established mathematical rules. For example, in this method the Euler-MacLaurin summation formula for approximating integrals by sums[1] is derived by taking the logarithm of a general real-valued function, with no regard to the negative values the function may assume (see, Friedman, 1991; pp. 176-178). It is only with the aid of functional analysis, which emerged early in the twentieth century, that mathematician were able to justify many of the operational method techniques.

This is an example of symbolic reasoning, a reasoning in which symbols are treated as if they possess a life of their own, and, accordingly, are manipulated without mental representations involving quantitative and spatial images.

Undoubtedly, symbolic reasoning plays a significant role in the development of mathematics. One might argue, for example, that the reconstruction of Calculus into Real Analysis at the beginning of the nineteenth century was chiefly a result of Fourier’s “symbolic solution” to

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[1] $\sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{f(0)+f(n)}{2} + \int_0^n (x-[x]-1/2)\, f'(x)dx$, where f has a continuous derivative on [0, n].

the flow of heat problem. As it is known, Fourier had reduced this problem to that of taking an even function and expressing it as an infinite sum of cosines, without attending the meaning of infinite summation of functions. His solution led to observations which seemed at the time inconsistent with "regular" behavior of functions.[2] This, in turn, led to thorough investigations into the assumptions of calculus and inspections of its structure, whereby the entire Calculus was reconstructed into a new mathematical field: Analysis (see Bressoud, 1994).

Apparently, symbolic reasoning has its origin in the nineteenth-century mathematics curricula as well. Consider, for example, Euler's textbook on multiplication of integers:

> Hitherto we have considered only positive numbers; and there can be no doubt, but that the products which we have seen arise are positive also: viz. +a by +b must necessarily give +ab. But we must separately examine what the multiplication of +a by -b, and of -a by -b will produce. Let us begin by multiplying -a by 3 or +3. Now, since -a may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is -3a. So if we multiply -a by +b, we shall obtain -ba, or, which the same thing, -ab. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down as a rule, that + by + makes + or plus; and on the contrary, + by -, or - by +, gives - or minus. It remains to resolve the case in which - is multiplied by -; or, for example, -a by -b. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign -, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign -; for -a by + b gives -ab, and -a by -b cannot produce the same result as -a by +b; but must produce a contrary result, that is to say, +ab; consequently, we have the following rule: - multiplied by - produces +, that is the same as + multiplied by +

[2] For example, when Fourier expansion of $f(x) = 1,\ -1 \le x \le 1$, is differentiated term by term, the resulted

Thus, Euler's rule for the sign of "- multiplied by -" was based on symbolic consistency needs rather than quantity-based considerations. In contrast, his derivation of the rule for the sign of "+ multiplied by - and - multiplied by +" was based on quantitative imagery representations.

Students reasoning symbolically

Despite the important role symbolic reasoning plays in mathematics research, its application in school mathematics, especially in the elementary mathematics curricula, can block students' mathematical development. As the following examples demonstrate, the most serious deficiency in our students' mathematical knowledge is the inability to reason mathematically about situations. In recent teaching experiments on the concept of mathematical proof (Harel and Sowder, in press), it was observed that students' reliance on symbolic reasoning is so ultimate that they are unable to attend the meaning of basic concepts they have learned previously. For example, students in their second course in linear algebra employed meaningless symbol manipulations to the problem "Prove $Null(A)\quad Null(BA)$" and were unable to express the statement "x is in $Null(A)$" algebraically (i.e., by the equality $Ax = 0$). Further, in many cases college students are unable to reason situationally even about elementary and basic concepts from high-school mathematics. The following example demonstrates this inability, and it, by no means, represents an exceptional event:

Patti's solution to a certain homework problem on limits of functions included the inequality $(x-1)^2 > 1$. Her solution to this inequality was $x > 1$. When she was asked to explain how she arrived at this solution, she responded:

> The solution to the equation $(x-1)(x-1) = 0$ is $x = 1$, x = 1 (She wrote down these three equalities).

Then she crossed out the three equality sings, wrote above them the inequality sign as follows:

$$(x-1)(x-1)\overset{>}{=}0 \text{ is } x\overset{>}{=}1,\ x\overset{>}{=}1,$$

series does not converge, which was against the mathematician's expectations.

and said:

x is greater than one.

Following this, Patti was asked to solve $(x-1)(x-1)=3$; she wrote:

$(x-1)=3, (x-1)=3$

Patti's mathematical behavior suggests that she was not thinking about the situations that these strings of symbols may represent; rather, the strings themselves were the situations she was reasoning about. That is, Patti's thinking was in terms of a symbolic, superficial structure shared by the three strings, not in terms of the quantitative, functional, or spatial relationships they may represent. From her perspective, these strings share the same symbolic structure and, therefore, the same solution method must be applicable to them all. Patti viewed each of these strings of symbols as a call for activating a certain procedure; namely, for her, the equality $(x-a)(x-b)=c$ calls for the application of the procedure $x-a=c,\ x-b=c,\quad x=c+a,\quad x=c+b$.

Because Patti's thinking was not based in situational images, she was unable to understand the teacher's view of these strings as representations of conditions that determine subsets of real numbers. Patti's conception of the problem at hand and its solution were, obviously, very different from what teachers usually assume they are communicating with their students.

Symbolic reasoning is a habit of mind students acquire during their school years, from elementary school to secondary and post secondary school. This habit of mind is very persistent and extremely difficult to relinquish. Below are solutions to a problem given at the end of a teaching experiment in which a major effort was made to reeducate the students to reason situationally. The problem was:

a and b are integers and c is an integer different from zero. Which of the following statement(s) is (are) true?

(i) $a=b \pmod m \qquad ac=bc \pmod m$.

(ii) If $a = b(\bmod m)$ $\quad \frac{a}{c} = \frac{b}{c}(\bmod m)$.

Karri's solution to (ii):

$\frac{a}{c}^{\cdot c} - \frac{b}{c}^{\cdot c} = (\bmod m)^{\cdot c}$ $\quad a - b = c(\bmod m)$. The only thing that changes are the number of m terms being multiplied.

Kathy's solution to (i):

True. If $a = b(\bmod m)$ then $ac = bc(\bmod m)$. From the statement above we know that a and b are congruent. So $a = x(\bmod m)$ and $b = x(\bmod m)$. Multiplying both by c, you get $ac = cx(\bmod m)$ [and] $bc = cx(\bmod m)$. Both are still congruent and can be written [as] $ac = bc(\bmod m)$.

Karri and Kathy's solutions are another example of a reasoning that is not based in situational images. Although these students realize that the content of these symbols is the domain of integers, they have not created a coherent network of quantitative relations and operations that correspond to the symbols and their manipulations. For these students, the symbol manipulation rules they acquired in their school years define the essence of their world of mathematical activities and mathematical truth. Image making and quantitative comprehension are virtually absent from their world of mathematics. Thus, the action of multiplying two sides of an equality by a number--a well practiced activity in algebra courses--constitutes the entire solution process applied by Karri; they raised no questions about the *meaning* of this rule in the context of module arithmetic.

Symbolic reasoning versus transformational reasoning:

The case of elementary Mathematics.

The dominance of symbolic reasoning in the elementary school mathematics curricula has well documented in the mathematics education literature. Students' error pattern in carrying out arithmetic operations is an example of the effect symbolic reasoning has on elementary-school students' mathematical behavior. However, symbolic reasoning is usually associated with manipulation with symbols in mathematical expressions, such as those we have seen earlier with college students, or such as the Cross Multiplication Formula (e.g., $\frac{4}{5} < \frac{5}{6}$ because $4 \bullet 6 < 5 \bullet 5$) commonly used by elementary school students. It is important to point out that symbolic reasoning is not restricted to this kind of symbol manipulation. Consider, for example, the Conservation Formula (Harel, 1995) commonly taught in elementary school before children build the conception that solutions of multiplicative problems are invariant under changes of the problem quantities. This Conservation Formula states:

When you encounter a word problem with "nasty" numbers,

(a) replace the "nasty" numbers with "friendly" numbers;

(b) solve the problem with the "friendly" numbers;

(c) transform back your solution to the problem with the "nasty" numbers.

For example, using the Conservation Formula, one can "solve" the problem "A cheese weighs 0.923 kg. 1 kg costs 27.50 kr. Find out the price of the cheese" by:

(a) replacing the multiplier, 0.923 kg, with a "friendly" multiplier, say 4 kg;

(b) solve the problem with the new multiplier by the operation 4x27.50;

(c) transform this operation into 0.923x27.50 (by replacing 4 by 0.923), which will be your solution for the original problem.

The mathematical behavior students adopts from learning solution strategies such as the Conservation Formula in elementary mathematics is consistent with the mathematical behavior we observe later with secondary and post secondary students. The Conservation Formula is an example of computation-centered school mathematics curricula, low demand for meaning and reasoning, and superficial considerations children employ in solving problems.

Transformational reasoning, in contrast to symbolic reasoning, involves operations on objects and anticipations of the operations' results. The operations are goal oriented, and they may be carried out for the purpose of leaving certain relationships unchanged, but when a change occurs, the observer not only anticipates it, but also knows what operation to apply to compensate for the change.

Against the Conservation Formula as a manifestation of symbolic reasoning, we bring an excerpt of an interview with two children, which demonstrates transformational reasoning in solving a multiplicative problem. This is an interview conducted simultaneously with two children, a 13-year-old girl, Tami, and an 8-year-old boy, Dan, and was reported and fully analyzed in Harel (1995). It demonstrates how children can reason trasformationally before they have learned that solutions of multiplicative problems are invariant under change of quantities.

Interviewer: One pound of candy cost $7. How much would 3 pounds of candy cost?

Tami: Three times seven, 21.

Dan: I agree, three times seven.

Interviewer: What if I changed the 3 into 0.31? What if the problem were: One pound of candy cost $7; how much would 0.31 of a pound cost?

Tami: The same. It is the same problem, you have just changed the number. 0.31 times 7.

Dan: No way! It isn't the same. Can't be (angrily). It isn't times. How did you (speaking to the interviewer) agree with her?

Interviewer: I didn't agree with her, I'm just listening to both of you. How would you solve the problem?

Dan: (After a short pause), you take 1 and you divide by 0.31. You take that number, whatever that number is, and you divide 7 by that number.

Dan's solution consisted of a plan for what should be executed to obtain the problem answer, and the entire executability of the plan was fully anticipated.

Characteristics of Algebraic Reasoning and The Effect of Symbolic Reasoning

We now focus on characteristics of algebraic reasoning and the consequences of symbolic reasoning to its development. In our view, two of the strongest characteristics of algebraic reasoning are:

a. The Quantity Representation Characteristic

The ability to think in terms of representations of measurements of quantities (not just in terms the measurements themselves) and in terms of representations of quantities (not just in terms the quantities themselves).

b. The Operative Algebraic Thought Characteristic

The ability to operate on the outcome of an operation without quantitatively evaluating the outcome.

We will discuss these characteristics in the context of the multiplicative conceptual field. Our basic point is that multiplicative reasoning serves as a corridor to and spring board for algebraic reasoning, in the sense of these two characteristics. For this, let us consider the concepts or ratio--the single most important concept that defines mutiplicativity.

Kaput and Maxwell-West (1995) and Thompson (1995) distinguished among several levels of ratio conceptions, a distinction that is not based upon situations but on the mental operations by which people constitute multiplicative situations. According to Thompson, for example, the three conceptual levels of ratio are as follows:

In the first level, ratio is where the multiplicative relationship is conceived as being between two specific, non-varying quantities. For example, as a ratio, a per-statement such as "3 cups of orange concentrate per 4 cups of water" is conceived as "a comparison of the two collections per se, or a comparison of one as measured by the other" (Thompson, 1994, p. ??).

In the second level, which Thompson calls "internalized ratio", ratio is where the result of the relationship is fixed as well as the quantities being related, but the values of the related quantities vary" (multiplicatively). Continuing with the same example, this per-statement would be conceived under this conception as a representative of all ratios between two collections of the same quantities--cups of orange concentrate and cups of water--where the values of the collections vary multiplicatively (e.g., from "3 cups of orange concentrate per 4 cups of water" to "6 cups of orange concentrate per 8 cups of water", etc.). Thus, a child who conceives, for example, the quantity of "taste" as an internalized-ratio would understand that a mixture with 40 ounces of water and 24 ounces of orange concentrate will taste the same as any other mixture with 40Xn ounces of water and 24Xn ounces of orange concentrate: By necessity, this child must think in terms a representation of quantity's measurements, rather than specific measurements.

Finally, in the third level, which Thompson calls "interiorized ratio", ratio is where the quantities themselves vary. For example, the ratio $a : b$ represents a multiplicative relationship between any measurements of any quantities.

This short analysis demonstrates how multiplicativity provides rich and natural context for developing algebraic reasoning, in the sense of the quantity representation characteristic. In what follows we will discuss how through multiplicative situations students learn to reason algebraically, in the sense of the operative algebraic thought characteristic. As an example of this ability, consider again Dan's solution. Recall this solution did not include any actual computation; it consisted only of a plan for what should be executed to obtain the problem answer. The plan being: "You take 1 and you divide by 0.31. You take that number, WHATEVER THAT NUMBER IS, and you divide 7 by that number." So Dan was able to operate on the outcome of the operation of dividing 1 by 0.31 without obtaining the actual value of the outcome. This way of thinking is indispensable in algebra, and constitutes the conceptual foundations for the concept of function.

As a teacher, I first observed the non-immediacy of the development of operative algebraic thought when I taught linear equations to junior-high school students. This observation is unlikely to be made if students are taught to solve equations symbolically (instrumentally, that is; such as, to solve $5x+12=17$, move the 12 to the right-hand side and divide by 5). I introduced the idea of solving an equation as a search for a number that satisfies the condition defined by the equation. In the process of teaching solution of equations, I noticed that with the exception of a few students, most of my students were unable to perform the intermediate stage of thinking of $5x$ as a number which when is added to 12 results in 17; they could search for x but not for $5x$. In my opinion, students' difficulty lies in their inability to operate (adding 12) on an outcome of an operation whose value has yet to be determined. This is an epistemological obstacle--a natural developmental difficulty, that is--that signifies the beginning conception of algebraic reasoning. If this obstacle is avoided by providing the students with a symbolic tool to solve equations, the birth of algebraic reasoning is likely to be postponed or killed altogether.

The idea of a "pattern" is another example where operative algebraic thought is required. To identify the general pattern of the sequence:

$$1 \rightarrow 1, \quad 2 \rightarrow 3 \quad 3 \rightarrow 5, \quad 4 \rightarrow 7, \quad 5 \rightarrow 9$$

and predict the outcome of any given integer, one must be able to perform the operations of multiplication $(2n)$ and subtraction $(2n-1)$ without carrying out the operations themselves. Ratio and proportion problems provide a natural context for developing this way of reasoning, for the notion of ratio, in the second and third levels discussed above, requires one to reason about the multiplicative relation between quantities without necessarily computing the relation itself.

Symbolic reasoning, we argue, deprives students from the opportunity to engage in these ways of reasoning about multiplicative problems and, therefore, blocks students from developing algebraic reasoning. This is so, because, as it has been established by Piaget and others, the main tool for modifying existing conceptions is true problem-solving activities, where the learner applies existing conceptions to solve the problems and modifies these conceptions when encountering cognitive conflicts. The use of the Conservation Formula and Cross Multiplication Formula, for example, sterilizes these conflicts and gives both the child and the teacher the illusion of accomplishment, where in fact the child is not experiencing any problematic situations that can bring her or him to invent ways of thinking multiplicatively.

Implications

We argue, therefore, that instructional activities that educate students to reason about situations in terms of transformations and compensations are crucial to students' mathematical development, and they must begin in an early age: kindergarten and first grade. Their reasoning at this stage is pre-notational unless invented by them. The following examples (brought up by the Quantitative Reasoning Group[3]) demonstrate the type of activities in early grades that can potentially promote reasoning about quantities and their relationships:

1. Your plant is taller than my plant. Your plant grew more (less; same) as mine. What could the situation be?
2. Allan is faster than Elliot in September. They both got faster. Who would win a race in December?
3. At table 1, 3 children share 3 cookies. At table 2, 4 children share 5 cookies. Is that fair?
4. Count how many of these small boxes it would take to fill the large box

<Insert Figure 1 ??>

- ∑ If you tried to fill the large box with small boxes that were twice as big (small), how many would it take?
- ∑ If you tried to use the small boxes to fill a larger box that was twice as big (small), how many would it take?

5. This scale is in balance.

[3] The members of the Quantitative Reasoning Group were James Kaput, Richard Lesh, Pat Thompson, Randy

<Insert Figure 2 ??>

- ∑ If you replaced the blocks on the right with a block twice as heavy, how many would it take to balance?
- ∑ If you replaced each block on the left with a block twice as heavy, and you replaced each block on the right with a block twice as heavy, what would happen?
- ∑ If you replaced the block on the left with a block twice as heavy, what would happen?

The notions of "heightens", "muchness", "fastness", "loudness", "fullness", and heaviness are examples of qualities. A quality in the empirical level is an action of actual experience, such as the feeling that results from walking fast, or the sense by which the flavor or savor of things is perceived when they are brought into contact with the tongue. At the operative level, quality is an *anticipatory scheme*--a scheme that leads to an answer before the details are filled in by the action during the actual process of arriving at it (Piaget, 1967). With this scheme, one compares between results of an experiential action without carrying out the actual action--a necessary mental operation in transformational reasoning. To reason about the problems described above one must have developed the qualities of height, fastness, heaviness, etc., at the operative level. Conversely, these type of activities are needed to develop this level of thought.

We conclude with what we believe two of the most important intellectual activities in reasoning about qualities: <u>quantification</u>--the process of assigning measures to qualities (see Thompson, 1994) and <u>geometrization</u>--the process of *spatial* imagery construction and organization. The importance of these processes is (at least) two-fold.

<u>Scientific reasoning.</u>

Through the quantification process students reason about <u>specific</u> quantities (speed, heat, etc.) and investigate relationships among them--an activity that constitutes the heart of physical sciences. It

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Philip, Judy Sowder, Ricardo Nemirosvsky, and myself.

is through quantitative reasoning that students build concept images of the specific quantities they reason about--images that are instrumental in understanding “theoretical” mathematical phenomena. As an example of this assertion, consider the quantity of speed in relation to the Fundamental Theorem of Calculus, and the quantity of heat in relation to properties of harmonic functions (most notably, the property that the extreme values of a harmonic function must be on the boundary of its domain). Similarly, spatial imageries--such as mutual positions of lines and planes in space--are indispensable to the understanding of linear algebra and multivariate calculus.

Imagery building for comprehension.

By reasoning quantitatively and geometrically, students learn the process of mathematics comprehension. That is, they learn that sense making and image building, not symbol manipulation, are the heart and sole of mathematical activity. For this, instructional activities must be designed for the purpose of fostering reflection and abstraction of the mathematical concepts we intend to teach our students. For example, multi-digit addition and subtraction problems should be introduced in a context that is experientially concrete to students--the context of money, for example. Students learn that in order to deal with the problems successfully, they must comprehend the situation quantitatively, and, in the absence of formal algorithms to solve these problem, they develop their own solutions. Their algorithms may be quite different from the standards ones, but when the latter are introduced, they would have the conceptual tools to comprehend them and make them their own.

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