Thinking in Variables

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The purposes of this chapter are three-fold. First, to describe and exemplify what it means to think in variables. Second, to describe a set of activities which have proven successful in helping a young precocious child become highly competent in algebra and to present, in some detail, his thinking as he constructed algebraic concepts. Thinking in variables is contrasted with learning procedures for solving equation types and transforming algebraic expressions. This paper suggests a promising approach to algebra which is quite different from algebra as defined by conventional albegra texts.

Kaput (This volume) lists five types of algebra. This chapter focuses on Kaput's types one and three which consider constructing patterns and relatins@ps and functions. During the past 24 months I have had the opportunity to conduct a teaching experiment with a mathematically precocious child. When the study began he was six years old. Because he has had few procedures imposed on him, it was possible to consider algebra from a meaningful rather than procedural perspective. In this chapter, I will describe an approach to algebra which emphasizes patterns, relationships, imaging, and problem solving. The concept of variable has been a focus of our work and extensive use was made of graphing calculators.

In this paper, I distinguish between thinking of a letter used as an unknown and as a variable. The operative word here is thinking, since the difference is in how one thinks about the use of a letter to symbolize a quantity. The concept of variable has received much attention (F,eggiani,1994; Schoenfeld and Arcavi, 1988; Usiskin, 1988; van Reeuwijk, 1995; Wagner, 1983; Wagner, and Kieran, 1989). In a general sense, n used as a variable or unknown is considered to stand for any number from a particular set. However, in the equation $3n + 2 = 14$, it is assumed that there is a specific number for which the equation is true. Thus thinking of n as an unknown can yield a solution and variable reasoning is not needed. In solving the equation $x2 - 7x + 12 = 0$ we are attempting to determine the number or numbers for which the equation is true, in this case a set containing two numbers. Again, x is thought of as an unknown. When we solve a word problem by saying "Let x = The number of centimeters", we are assuming there is a particular number of centimeters. Again, using a letter as an unknown may be all that is needed. To say "Let x=' the unknown, writing an equation and solving it can be a powerful technique for solving problems which can be so formulated but is does not necessarily require thinking in variables.

Learning to think in terms of variables is a major advance in doing mathematics. In variable reasoning, a letter is used to symbolize any number from the domain without singling out any particular elements of the set for special consideration. For example, when we write 2n + 1 for an odd number we are asserting that 2n + 1 is odd, n any whole number. A relationship is being expressed rather than solving for an unknown. If we wish to express the relationship between the cost in dollars of mailing a letter and its weight in ounces, we can use n for the number of ounces and write C = .3 2 +.23(n- 1). In this mathematical statement, we are thinking of n as any natural number (but recognizing there is some upper limit). The use of n as a variable here is quite different from thinking of n as an unknown number in the equation 3n + 2 = 14. Mason (This volume) suggests that this manifest ability to work with unspecified numbers is one of the key skills of arithmetic that are needed for algebra. We can say a student has learned to think in variables when s/he spontaneously generates a number sentence or expression symbolizing a relationship embodied in written or graphical form. It is one thing for a student to write an equation when asked to translate a sentence. It is quite another thing to choose to express a pattern or relationship in algebraic notation without prompting. Being able to write equations and manipulate expressions does not always indicate that the student is thinking in variables.

Thinking in variables is analogous to thinking in Venn diagrams. When solving a problem, we may choose to express the relationship of quantities using Venn diagrams. When a student initiates the use of Venn diagrams in problem solving, we can say the person has a new way of conceptualizing the relationship between quantities. Thinking in Venn diagrams is to be distinguished from being able to answer questions expressed in Venn diagrams. In a similar sense, using variables in formulating mathematical relationships is like using pronouns to state a relationship between people, e.g., "they all like angel food cake."

With the heavy emphasis elementary algebra texts place on letters as unknowns, few high school students learn to think in variables. The extensive repetitive exercises solving linear equations by types using prescribed methods does not encourage variable thinking if it encourages any thinking at all - it aims to teach procedures. High school algebra texts usually have pages of practice solving equations, first of the type ax = b and then ax + b = c before moving to equations with letters on both sides of the equals sign. I use the word letters because in this setting the students are required to uses a prescribed procedure so the lessons are about methods rather than concepts. When students are computing with expressions, for example, multiplying polynomials, the letters used are not thought of as variables or unknowns; they are just symbols which are part of the prescribed procedure. That is, sense making is not on the agenda. But it is on the reform agenda.

One student said, "When we find what x is, let's write it down so other people won't have to look for it.'

Michael's Early AIL-ebra

It was clear in the first two weeks of the study that Michael (age: 6 years 0 months) had constructed the concept of a letter standing for an unknown number. For example, he could inunediately state the number which when substituted for n in n + 5 = 12 would make the sentence true. However he had not yet learned to think in variables.

Based on an initial evaluation of Michael's mathematical knowledge, a series of activities were planned to help him develop algebraic reasoning. Some problems were posed in a balance format as shown in figure x

Insert Figure 1

We played What's My Rule? at which Michael excelled. When it came to his turn to make up a rule it was "Multiply the number by itself and add two times this amount (algebraically, nxn + 2nxn). He also solved equations of the form 60+n=4n.

A significant construction occurred during our sixth session, age 6: 1. This activity was designed to help Michael use variables in his mathematical reasoning. While he had used letters for unknowns and could solve n - 6 = 17 with ease, he did not, at this time, think in variables.

I prepared envelopes of several weights and provided a postal weighing scale. I handed Michael an envelope and asked him to determine how much postage was needed to mail it first class. We negotiated cost structure used by the postal service and he easily determined the postage needed for a regular letter and an envelope weighing between six and seven ounces. Then the following exchange occurred.

Postal Scale Problem. Age: six years, 1 month (6:1), Sixth session,

I present an envelope and a postal scale and ask how much postage it will take to mail the letter. He puts it on the scale and we talk about how the Post Office works. He figures 52 cents for first envelope (29 + 23).

W: We have a bigger package. So we want to know how much postage to put on this one (handing him a larger envelope).

M:

W: Oh no! (looks at scale indicator) About 6 V2 ounces. OK. The post Office doesn't deal in part ounces. What are we going to use?

M: (Gets a calculator, has decided to use seven, says nothing but arrives at an answer using the calculator.)

W: So how many ounces did you use? (pause) Tell me what you did.

M: First I went, like 6 x 23 + 29 100.

M: (inaudible)

W: Remember what you said?

M: $1.67 (W writes it on the board)

W: For how many ounces?

M: Seven (W writes 7).

W: Seven ounces cost $1.67.

M: That time it worked.

W: Now suppose we worked in the Post Office and anytime anyone came in we wanted to have a quick way of deciding how much to charge them. Is there some way we could write this down? Let's write down what we did, maybe you could write it on the board, just what you did to figure it out and then maybe we can do every time.

M: First (writes 7 oz. and below it 6 x 23 + 29 100).

W: Now Michael, is this a plain 23, plain 29? and then when you are finished you divide by 100?

M: So instead of it being 150 DOLLARS (laughs) ... That would be four or five POUNDS.

W: I wouldn't want to have to pay 150 DOLLARS!

M: But you would pay $1.67.

W: Now this time I'm bringing m a package, let me say it weighs, oh.... x ounces. (I write x ounces.)

M: What's x?

W: How many ounces.

M: Hum. . .

W: Can you just write out what we would do now?

M: First. . . I have to go ... (writes - x.23 + 29/ 100=) That's the ... (points to the - -)... I'll have to figure out that.

W: My question is, what goes in your blank there at the front?

M: "Don't.. have it ... yet."

W: OK. I noticed you wrote 7 oz. up there but used six. *Where* . . .

M: That's because ... that's (pointing to 29 in his symbolization) so that means (writes 7 - 1 = 6 vertically).

W: OK. This time instead of 7 we have x.

M: Right and that means we don't have it yet.

W: OK. So if it happened to be 13 ounces what would you do?

M: (writes 12)

W: Now how do you get 12?

M: Just like that (Writing 13 - 1 = 12 vertically).

W: I have a suggestion. What about if we were to write 13-1 in that place. It might help us later on. Let's write 13 - 1. Now write 13 - I there and put parentheses around it so we will know. . . all right so whatever one number they give us ... say, x ... Like if x happened to be …

M: Well the first one was…

M: Yours would be $3.05.

In this session "We don't have it yet' suggests that he had no prior experience symbolizing thought using a letter as a variable. In fact, writing (x - 1) times .23 + .29 as an expression for an anticipated action is quite sophisticated. In the very next session I presented the following task which had the same structure as the postal problem. In 29 seconds, Michael reported the correct answer. This confirmed the observation that he had conceptualized the previous problem in abstract terms.

My son lives in Philadelphia and the other day I called him and it cost $1.08 for the first minute and $.28 for each additional minute. If the total charge was $8.36, how long did we talk?

One year later, I posed a generalization of the following problem which he had solved in the previous session (The new statement used s for a side and 3s for length of string).

Insert Figure 2

<u>(Age: 8:3)</u>

W: We've got an equilateral triangle with side of length s and this segment is of length 3s.
M: You mean like last time, that 3, 3, 3, and 9?
W: Yes but this time it is s, s, s, and 3s. Now could we state exactly in terms of s the area of these three regions (Pointing to the three sectors drawn around an equilateral triangle)? (interruption)
W: OK. So this is our problem. We want to describe the area swept out by this segment. . .
\- - - -
M: What is s equal to in number?
W: We don't know. It's any number ... see, once we figure it out --afterwards anybody can tell us a number and we can just plug that value in for s and we will already have figured out what it would be - the problem is sort of solved --- someone tells us s and we plug it in.
M: s might be a huge - a hard number like 1,974.
W: Yeah.
M: That would be - - (done?) on a calculator - ten, that would be simple!
W: It could also be a number like ...
M: 5.82?

W: Yes, or eight times the square root of two all over 1964832796 - be any number (M laughs).

Michael asked, "What is s equal to in number?" But he quickly indicates he understood that s was acting as a variable. This question, 'What is s?" at this juncture in our study was quite significant. Michael was reflecting on the use of a variable. In writing an expression for area in terms of s is to use a letter as a variable, not just an unknown. Perhaps we should consider x as an unknown to be a special case of x as a variable.

At age 8:2 the following problem was presented:

> The cost of renting a car for a day is $24.95 plus 12 cents per mile driven. Write an equation which gives the cost *y* (in dollars) in terms of the miles driven *x*.

In five seconds, Michael wrote 24.95 +.12x. He then added y= in front of the expression. It is clear from the video segment that as soon as he read the problem, he conceptualized the relationship and symbolized it using algebraic notation. Initially, his focus was on 24.95 + .12x as a quantity rather than writing an equation. This provides strong evidence that Michael had developed the concept of variable and could symbolize his thinking using algebraic notation. Notice that this activity does not involve considering x as an unknown since we are not solving for x but using it as a variable. As we were discussing possible interpretations of the problem, Michael excitedly said, "Oh, I can graph it." He proceeded to enter the equation in his TI 85 and began exploring the graph by adjusting the range, zooming and boxing.

At age 8:4, I posed the following problem:

> A building is constructed with a 30 inch foundation and each brick with one layer of mortar is 4 inches. Write an equation which expresses the height of the building in terms of number of bricks.

Michael's first question was "What is mortar?" After this was discussed he mmediately wrote h = 30 + 4x. This is strong evidence that Michael has learned to think in terms of variables.

At age 8:4 the following problem was posed:

If one plant will grow to fill a lawn in 30 days, would two plants fill the lawn in 15 days?

In comparing 2^{30}, 2^{15}, Michael set up a table and labeled the colunms 2^x and $2(2^x)$. He was not using 'x' as an unknown - he was not attempting to determine THE value of x but used it to symbolize a functional relationship. He began writing values for 2^x and $2(2^x)$ in numerical form, e.g., 1, 2, 4, 8, 16,. . . Later, he wrote and graphed $2(2^x)$, $3(2^x)$, $4(2^x)$, and $5(2^x)$. This is additional evidence of thinking in variables.

When will y out-number x?

Much of Michael's mathematical reasoning is image-based in the sense that he used image schemata as described by Johnson (1987) rather than just mental pictures. For Michael, algebraic notation was more than st@gs of symbols to be manipulated - it frequently evoked images of graphs. However, his imaging was usually of an abstract nature (schemes) rather than images of particular materials or objects. He looked at a set of symbols and instantly constructed a mental image of a relationship (as most mathematically soplfisticated persons do). For example, at 7:11 he looked at y = x2 - 5x and immediately "saw" a parabola. He indicated this by a sweeping of his arms, verbal description, and a sketch of the curve.

Insert Figure 3

W: When will . . .
M: A little earlier.
W: What do you mean?
M: Well it will be negative 4 when x is one. It DO@. (points toward the floor.) At five ... zzzero.
W: What about after five? What is going to happen?
M: Itll go ... be positive but will never be . . . but you see when you get higher up it still is over (y > x)...
W: What do you mean by over?
M: Over x. But when x stops outnumbering . . . I mean you start at zero and go over, when will x be out-numbered?
W: Explain what you mean by that.
M: At zero however ... here's zero (drawing) keeps going over til its out-numbered.
W: Out-numbered by what? Now at zero what do we have?
M: (Shifts his attention to the origin.) zero and zero. Then at 1, -4 now y is out-numbered.
W: What do you mean by out-numbered?
M: x is out-numbered. Ah, so it's actually a parabola, zero, five and its

lowest point is 2.5 for x. (With his two hands, one for each branch of the parabola, he starts up high and sweeps down and together tracing in the air a parabolic shape. It is significant that he brings them together; enacting symmetry and acknowledging a lowest point.
W: What do you think its lowest point will be?
M: Seven. That's only a prediction.
(Tmmediately picks up TI 85 and evaluates $x^2 - 5x$ at $x = 2.5$

As he considered $y = x^2 - 5x$, he asked the question, when does y outnumber x? This question, "When does y out-number x?" suggests that he had constructed a dynamic relationship between two variables, x and y, and he was considering the interplay. The use of the term 'out-number'is noteworthy. This was Michael's language since the phrase had not been used by any of us in previous sessions. His question also suggested that he realized that in certain ranges, x was larger and at other times y was larger. Further, it indicated that he knew that as x became large, y would increase faster (rate of growth). Since he had constructed an image of a parabola associated with equations of the form $y = ax^2 + bx = c$, it also indicated that he first focused on the right half of the curve, knowing that the other side would mirror it. His image of the graph of $y = x^2 - 5x$ was more than a picture in his nd of a parabola. He spontaneously focused on the interplay between two variables. He was thinking holistically rather than sequentially. There is a sense in which he conceptualized a dynamic relationship between x and y. He obviously has constructed the coordinate plane (image-based) as a symbol system useful in thinking about interrelationships. After trying only two values, (1, -4) and (5, 0), he confidently announced that when x became greater than 6, y would 'out number' x. This episode is a clear example of thinking in variables. The meaning Michael gave to the set of symbols, $y = x^2 - 5x$, was of two quantities changing in relation to each other. His thinking was heavily influence by extensive use of graphing calculators. When he looked at an equation, he naturally thought about its graph which showed how the two variables were related and changed together.

His first question was, "When will y out-number x?" Quite remarkable, he quickly answered his own question, noting that at 6, x and y were the same. He then "checked" his assertion by using a calculator to evaluate $x^2 - 5x$ for x = 6.01.

Exploring graphs

The graphing calculator greatly influenced Michael’s thinking. Since the TI85 requires an equation in two variables to be expressed in the $y = f(x)$ form, it forgrounded that equation format. More importantly, extensive graphing of polynomial and other algebraic functions influenced Michael to form a mental image of the graphs of algebraic equations. That is, he came to anticipate the apearance of the graph when considering an algebraic equation. He also came to consider the contribution of each part, e. g., term, of an equation to the associated graph. But above all this was the development of joint variation of variables (Kaput's type 4). On several occaisions michael focused on the interplay of two variables, e. g., “When will y out-number x?" This is clearly a more sophisticated form of thinking in variables, one which goes beyond considering changes in the dependent variable resulting from change in the independent variable to thinking of the two variables jointly varying, more sophisticated than using a letter to express the nth term of a sequence. In writing the nth term of a sequence, e. g., $3n + 7$, the expression is the focus. The fact that there is a functional relationship and that n takes on the particular values 1, 2, 3,. . .n, is very much in the background and the interplay of n and $3n + 7$ is not considered. While this is certainly an example of thinking in variables, only one variable is actually symbolized. In contgrast, the equation $x + y = 12$ can evoke joint variation; as x increases y decreases and vice versa. When Michael asked, "When will y out-number x?" he was conceptualizing an inter-dependent changing of x and y. In the x, y pair, for some values x was larger than y and for other values, y was larger than x.

On previous occasions Michael and I had considered functions and their graphs and had negotiated a culture of exploration which included looking for maxima, minima, roots and symmetry. As the following episode suggests, exploring the nature of a graph can be a powerful stimulus for mathematical reasoning. In the process of exploring a quadratic equation, Michael showed the depth of his mathematics knowledge, his curiosity about number relationships, an intention to make sense, and the power of his mathematical reasoning. It is obvious that he was thinking in variables.

At age 8:7, Michael initiated the determination of the largest value of y in $y = 2 + 5x - x^2$. Raising this question is an indication of his intellectual curiosity. He just wanted to know the highest point period, not to be used in any particular way. He frequently pursued such issues. When I posed the initial question of this episode, Michael was holding a graphing calculator and could have easily just entered the equation in his calculator but chose to explore it before hand. I had asked, "What do you think it would look like." Michael would often ignore my comments or questions if he was engaged in another line of thought.

Insert Figure 4

W: Suppose we wanted to look at $y = 2 + 5x - x^2$.
(Michael is still exploring a complex graph he had generated.)
W: Would you put that equation in please? (He ignores me)
M: First of all ...
W: What's it going to look like?
M: (softly, $2 + 5x - x^2$)
(M begins making a sketch, drawing axes and putting appropriate hash marks and sketching the curve. He initially sketched it opening downward with the vertex at (0,2))
M: $+5x$ will . . .
W: Let's graph it to see how close you came (He doesn't).
M: 2.5 is the highest we will need.
W: OK,. . . 2.5?
M: Oopsy, I know what's wrong.
W: I don't know how you figured 2.5. OK.
M: Umm, 7.5.
W: How did you decide that?
M: Took an estimate. Because at first I thought it kept on going downward [from (0,2)] but then I realized that when you add the $5x$, that $+5x$
W: Why did you change it from 2.5 to 7.5?
(Michael erases and redraws the graph symmetric to $x = 2\frac{1}{2}$)
M: I realized when you added, . . . that $5x$, puts it up (hand motion) but ONLY until it and then meets it ... right there! [meaning along the horizontal line through (0,2)] (Plotting (5,2).
W:
M: (puzzled) What is that? At 5? What is this across ...
Five two, zero two [(5, 2) (0, 2)]
W: (5, 2)?
M: Right there is (0, 2)
W: What is that?
M: And you see they meet because x times itself... but $5x - x^2$ at 5 is zip (meaning 0). You see, $5x$ at 5 is 25. You have 25 and 2 but then you have to take away ... those don't have anything.

[My conjecture is that he quickly identified the point which would pair with (0,2) on the curve. His intuition (slippery word) led him to quickly identify (5,2) - he tried no other values before naming it.]

W: OK So these [(0,2), (5,2)] are on the same line. The axis of symmetry.
. . since these two points are on the same line, I say 2 1/2 is where the highest point is.

(M begins entering equation in calculator).

M: x is that line (points to x-axis. He is deciding on the range of x and y to enter in calculator). - 10 to 7.5 (for y range).

W: Let's. . . - 10 to 10, OK, - 1 0 to 7.5.

M: We could see it bump the axis.

(The equation is graphed and we both look at it)

Oooh ... just a little . . . A little off, not bad.

W: Excellent

M: Not a very bad guess.

W: I think if was a wonderful guess.

W: Michael, there is this new function on the calculator I want to show you called SHADE...

(Michael ignores my comment because he is still thinking about the quadratic function)

M: Let's see what the max actually is. (begins mentally evaluating the polynomial for $x = 2\frac{1}{2}$).
8 1/4.

(Later as we were attemmpting to use the shade function of the TI-85, he announced that the curve crosses the x-axes between 0 and 1 but clearly means 0 and -1. He estimates -.75.)

Michael looked at the quadratic equation $y = 2 + 5x - x^2$ and constructed an image of the curve in the Cartesian plane. Rather than just seeing letters and numerals, an image was evoked. It was Michael's intention to make sense that generated this response. Further, his imaging proceeded recursively as he reflected on the meaning of the equation. He first drew a sketch of his initial image which was influenced by his prototypical image of a quadratic equation as a parabola. Because the coefficient of x^2 was negative, he drew the parabola opening downward but symmetrical to the y-axis. The vertex was placed at (0,2) because of the constant term 2 in the equation. As he considered the equation and his sketch, he refined his image of the graph, realizing now that the $+5x$ term would shift the graph up and to the right. He then changed his estimate of the highest value of y to 7.5. He plotted (5,2) and considered it a special point, - the point on a horizontal line with (0,2), using the symmetry of the graph. At this time he graphed the equation on a TI-85 calculator. We then discussed where the axis of symmetry would be and concluded it would be at $x = 2\frac{1}{2}$. He then mentally evaluated $f(x)$ at $x = 2\frac{1}{2}$ and obtained 8 1/4 as the largest value of y. He was now satisfied with the graph. I pointed out that we did not yet know where the graph intersected the x-axis. He commented that one of the roots would be between 0 and -1, observing that for $x = -1$, y would be negative because the middle term would dominate and be negative. He had already noted that y was 2 (positive) when x was zero so the curve had to pass from negative to positive and thus intersect the x-axis between -1 and 0.

When Michael looked at $y = 2 + 5x - x^2$ he constructed an image of a parabola opening downward with vertex at (0,2). As he thought more about this function, he realized he had not incorporated the effect of the $+5x$ term, so he adjusted his graph, moving it up and to the right. As he did this, he used symmetry and plotted the point (5,O) as a "mate" for (0,2), a point on the curve. Imbedded in his thinking was the recognition that the parabola had an axis of symmetry, that it would ‘come down’ the same way it went up on the left. That is how he came to single out (5, 2) as a significant point.

Task (Age - 8:11):

Dixon Middle School has two seventh grade classes in rooms 1 and 2, and two eighth grade calsses in rooms 3 and 4. Rooms 1 and 2 together have 50 students and rooms 3 and 4 together have 46 students. Room 1 has six more students than room 4 and room 2 has two fewer students than room 3. How many students can be in each room? Is there only one possible class size?

M: (Reads problem out loud.)

W: OK …

M: First, (slight pause) Let's call room 1 “x” and room 2 “y” (writes)

Room 1 = x

Room 2 = y

Room 4 = $x - 6z$

Room 3 = $y + 2w$

$$x + y = 50$$

$$w + z = 46$$

W: What are we going to do now?

M: First, (slight pause) Let's see what happens here (makes a table with headings x, y, z, w.)

W: Ah, good.

M: $x = 25$, $y = 25$, $19(z)$, $27(w)$

W: So that's . . .

M: But I will put ... (makes two more columns at the right labeled 4 check and X to indicate whether all conditions were satisfied.). Check! (Makes a check mark for the row with $x = 25$,) $x = 26$.

W: So that a possibility ...

M: Let's see what happens when $x = 26$. (without hesitation or pause, writes 23 (y), 25, 21

Check!

M: $x = 27$

W: What are the limits?

M: x cannot be any larger than 49

W: What about 49?

M: Yes. Let's try it

W: So what is smallest?

M: X cannot be any lower than 7

W: And what about all the numbers in between?

M: Check!

W: So how many solutions will that be?

M: 42.

W: Ah, I don,t ...

M: No, 43!

In considering this problem, Michael immediately and withouthesitation, chose variables, named them and wrote down four equations which symbolized the information. He then proceeded to explore the range of possible values for the variables using a chart which proved to be both effective and efficient in solving the problem. He tried 25, 26, and 27 for x, at which time W asked, "What is the largest?" He responed by saying, "Any number up to 49." W then asked, "What about 49?" Michael said that 49 works and it is the largest. W then asked about the smallest to which Michael responded seven, and proceeded to show that $x = 7$ worked and than any smaller value would not. When asked how many solutions there were in all, he responded 42 (49 - 7) but quickly changed that to 43. The entire solution time was less than five minutes.

In this last instance of Michael's activity there was clear evidence of thinking in variables. He did not just try values in a guess & test manner or write equations and proceed to manipulate them; he did not smft into an algebraic computational mode, even though he did many numerical compputations. This problem had many solutions so it was not a matter of solving for an unknown - the letters were used as variables. Thinking in variables allowed him to formulate this problem meaningfully and efficiently. This power will prove useful as he attempts to conceptualize and formuate relationships in many settings.

CONCLUSION

In this chapter, an attempt has been made to describe an aspect of mathematical reasoning called thinking in variables. Evidence of thinking in variables would be the spontaneously generation of algebraic expressions which symbolize a mathematical relationship based on an experience such as reading a nonroutine word problem. While students can be trained to translate certain types of word problems into algebraic equations, thinking in variables goes well beyond translation. It is characterized by being able to anticipate an algebraic symbolization of a mathematical relationship. For example, Michael read a problem involving the angles of a trapezoid being in an arithmetic sequence and wrote

$$x + (x + d) + (x + 2d) + (x + 4d) = 360$$

He was able to think of the measure of the smallest angle and the constant difference as variables which he then expressed symbolically.

Too often, algebra students fail to learn to use variables in a meaningful way (Schoenfeld and Arcavi, 1988; Usiskin, 1988). In this chapter, I have attempted to provide detailed evidence of one student's thinking in variables. A key indicator of thinking in variables is the spontaneous generation of written or number sentences using letters to symbolize relationships expressed in numbers, words or pictures. This is one component of what Arcavi (1994) calls symbol sense. It is not enough for students to use letters as unknowns and manipulate equations and expressions using letters. On many documented occasions, Michael formulated equations or expressions to symbolize relationships. This spontaneous symbolization was taken as evidence that he was thinking in variables. Michael learned to think in variables, in large part, because of the particular activities in which he was engaged. The set of activities described in this paper provide rich opportunities for students to make important algebraic constructions.

Most of the work in schools on functions tends to be notational, mechanical and not always meaningful to students. It is profoundly important that students learn to think in variables and express relationships in algebraic notation - this is mathematics. It was quite clear that after extensive activities thinking in variables, the function notation was easily given rich meaning - there were "hooks' to hang the symbols on. When $y = f(x)$ is used in introducing functions, students have many things to attend to and the symbolism does not always evoke functional concepts. However, when students have been graphing $y = x^3 - 3x^2 + 10x$ and thinking in terms of the relationship between x and y, $f(x)$ becomes simply a convenient way of symbolizing an expression in x. Using functional notation or even defining functions is easily accomplished *after* attention to variables. However if we attempt to 'teach' functions first it may have little meaning to the students. Thinking in variables should be a major goal of school mathematics programs. The meaning of functions follows easily when students can think in variables. It could be argued that learning to think in variables is learning functions.

An examination of the typical high school or college algebra text will show that to most students, algebra is the manipulation of symbols. Algebra courses are designed based on the belief that if a student learns to factor, simplify expressions and solve classes of equations they will be able to do mathematics. Word problems are often among the most difficult topics for students. Many students say, 'I can't do word problems.' When a student learns to think in variables, 'word problems' are viewed in quite a different light.

Mathematics is obscurred by teaching procedures (NCTM, 1989). If we reconceptualize algebra as the activitiy of constructing patterns and relationships, students can become mathematically powerful. Using the Standards (NCTM, 1989) as a guide, the Connected Math Project, CORE PLUS and Interactive Mathematics Project are examples of curricula which attempt to do this. Competency in using algebraic notation is best developed by providing opportunities for students to think in variables in meaningful settings rather than drill and practice. Algebra is not just transforming expressions and solving equations

Problem centered learning is quite effective in learning mathematics (Wheatley, 1991). If an individual learns to think in variables it is *their* accomplishment. While a teacher may play a critical role in designing tasks and negotiating the classroom environment, in the end it is the individual that acts to construct. By presenting well designed tasks to individuals and groups of individuals, the teacher can facilitate thinking in variables. Explaining particular ways to solve problems or notate ideas is not always helpful. There is the danger that the student will take the method shown as THE way to do the problem and attempt to appy a recipe.

Learning to think in variables occurs over a long period of time. In large part because of behaviorism, we tend to think we 'teach a concept" in a particular unit of study. Thinking in variables requires considerable mental reorganizations and requires a variety of experiences over months and even years. Coordinated schemes and schemes of schemes must be constructed. Just as a landscape architech thinks in terms of trees, shurbs, and flowers when planning a yard design, a mathematician thinks in terms of variables in conceptualizing the landscape of a mathematics problem. This capacity was not achieved in days or weeks but over a much more extensive period of time, even for this very special student. As Anghileri (1995, p. 10) states, "Children need to develop flexibility in their interpretation of words and symbols and to associate with them different meanings and procedures for the solution of different problems."

This chapter draws heavily on research with a highly motivated and intelligent student - a student who had not become procedurally oriented in his thinking. What does this have to say about typical students'learning of algebra? First, the particular tasks and the intstructional strategies of the research project suggest alternative ways for all students to construct their mathematics. Some students will require more time than others but the focus on thinking in variables can be beneficial to all students. How students acquire algebraic computational competence was not addressed directly in this study. However, Michael became quite good at using and transfroming algebraic expressions without directly practicing specific procedures. Some students may need tasks designed to foster computational competency but a problem centered approach seems promising for all students.

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